PCTM Magazine

An Official Publication of the Pennsylvania Council of Teachers of Mathematics

Cover: Shea's Performing Arts Center in Buffalo, NY. Photo courtesy of Christopher Williams.

FRINGE THOUGHT: Learn something new with an old friend.

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Volume LVI

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The **PCTM Magazine** editors are:
Cynthia Taylor and Tyrone Washington (Millersville University)

President's Message

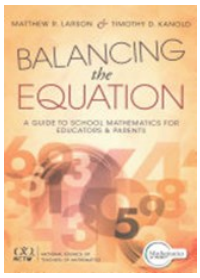
Lynn Columba

Dear PCTM Membership:

PCTM conference attendees are passionate about teaching and learning mathematics and are open to learning new concepts and activities for their students. Just as home classrooms provide a warm and welcoming climate, the 66th Annual PCTM conference at the Hilton Harrisburg created that same friendly atmosphere where many mathematics educators gathered to chart a course for the upcoming school year. Participants shared the belief that ALL children can excel in mathematics. Teachers, who are committed to their professional growth, will help their students to learn and succeed in mathematics.

> *A leader takes people where they want to go. A great leader takes people where they don't necessarily want to go, but ought to be.*--Rosalynn Carter

As reflected in the above quote, leaders in mathematics education shared their words of wisdom. Nationally recognized mathematics leaders in the field, keynote speakers and a special guest speaker, were: Dr. Matt Larson, National Council of Teachers of Mathematics (NCTM) President; Christopher Brueningsen, Headmaster, Kiski School; and, Pedro A. Rivera, Pennsylvania Secretary of Education.



Keynote Speaker on Wednesday—Dr. Matt Larson, NCTM President, who was the K–12 curriculum specialist for mathematics in Lincoln (Nebraska) Public Schools for more than 20 years. Dr. Larson began his career in education as a high school mathematics teacher and served as a member of the leadership team for the National Science Foundation Math and Science Partnership project Math in the Middle at the University of Nebraska–Lincoln. Dr. Larson's long history of service within NCTM includes chairing the Research Committee, serving for three years on the Board of Directors, serving for two years on the Executive Committee, and chairing the Budget and Finance Committee. Dr. Larson discussed **A Brief History of Mathematics: Lesson for Today.** The handout from his presentation can be found at http://www.nctm.org/uploadedFiles/About/President,_Board_and_Committees/Board_Materials/MLarson-SF-NCTM-4-16.pdf.

One of Matt Larson's publications, coauthored with Timothy D. Kanold, *Balancing the Equation: A Guide to School Mathematics for Educators and Parents,* is intended for educators and parents who seek to improve their own understanding of mathematics teaching and learning to help students succeed. The authors tackle popular misconceptions and misguided discourse about mathematics education and draw on peer-reviewed research about instruction that can significantly improve student learning.

Keynote Speaker on Thursday— Christopher Brueningsen, Headmaster of the Kiski School, a private all-boys residential college-preparatory boarding school near Pittsburgh, spoke about **The Changing Landscape of Math Education.** World markets continue to shift away from industrial economies based on production of goods toward "knowledge economies" that trade in information. Conference attendees explored the changing landscape of mathematics education in the U.S. and examined new teaching paradigms that encourage the development of skills that are most relevant for today's educated workforce.

Special Guest Speaker on Thursday— Pedro A. Rivera, Pennsylvania Secretary of Education, spoke on the topic, **Promoting Equity in Education Across Pennsylvania.** During his talk, he shared highlights regarding key initiatives that PDE is engaged in to promote education equity throughout the state. Also, he discussed the highlights of Governor Wolf's historic investment in education, updates from the Every Student Succeeds Act (ESSA) plan, and emphasized the importance of STEM integration across all key initiatives.

Many exciting sessions for elementary, middle, high school and higher education levels focused on a variety of topics including gaming, robots, WebQuest, logical reasoning, fractions, calculus, STEM, STEAM, and numerous others. Twenty-four vendors shared their resources for current mathematics lesson ideas for classrooms PreK-12.

As you plan for your school year and your professional growth, I invite the PCTM community to join in a year of growing and learning together. Next year, the 67th Annual PCTM conference will be at the Hilton Harrisburg, August 6-7, 2018. Mark your calendars! See you then!

Mathematically yours,
Lynn Columba, PCTM President

FRINGE THOUGHT: A wishbone is no substitute for a backbone.

2017 Annual PCTM Summer Conference Summary

Edel Reilly

On behalf of the 2017 Annual Conference committee, I would like to thank all of you who attended the 66th Annual Summer Conference at the Hilton Harrisburg on August 2-3rd 2017. I do hope you managed to attend some great sessions, meet some new people, and find lots of ideas to take back to your classrooms. It was great to see so many first attendees and I certainly hope you plan to return to our conference next year!

On the first day of the conference, we had close to 40 breakout sessions that covered some excellent topics including *The First 10 Minutes of Class*, *Strategies to Engage Learners in Mathematics,* and *A Preparation workshop for the Keystone Algebra Exam*. The highlight on the first day was the keynote talk by NCTM President Matt Larson. Dr. Larson's talk entitled *A Brief History of Mathematics: Lessons for Today* showed us that the issues and arguments on how mathematics is taught today are the same as those since the founding of the country.

The second day of the conference gave us close to 30 breakout sessions where many participants learned *Alternatives to Worksheets, Tips and Tricks on the TI-84,* and *Dynamic Desmos in the Classroom* to name a few. Our keynote speaker on Thursday was Mr. Christopher Brueningsen, Headmaster and mathematics teacher who spoke on *The Changing Landscape of Math Education.* Mr. Brueningsen shared new teaching paradigms that encourage the development of skills that are relevant for today's educated workforce.

We were also fortunate to have Secretary Pedro A. Rivera as a special guest speaker. Secretary Rivera shared ways that PDE is working to *Promote Equity in Education Across Pennsylvania*. Several initiatives were shared including Governor Wolf's investment in education, Pennsylvania's Every Student Succeeds Act, and the importance of STEM integration.

Many people helped bring this conference to fruition and I would like to take this opportunity to thank them. **Cynthia Taylor** and **Tyrone Washington** as Program Co-Chairs have worked tirelessly since the last conference to put together a wonderful program. There was a wide variety of sessions to choose from that ranged from early childhood mathematics to higher education. **Leslie Frischman** did an outstanding job as Exhibits Chair. We had over 26 exhibitors who were excited to demonstrate their wares. Many donations were given by exhibitors and individuals through Leslie's hard work that provided for nice prizes during our several raffles throughout the conference. **Annette Cook** served in the role of Registration Chair making sure everyone was able to participate in the conference. Annette also organized a team of workers to stuff the registration materials that were given to each attendee. **Steve Cicioni** served as the Treasurer making sure the bills were paid and that we stayed on good terms with all.

Christine Weigand worked with NCTM to have available for conference attendees many of the great publications NCTM currently has available for purchase. **Tim Seiber** clocked up many, many steps as Technology Chair walking from room to room making sure the projectors were in working order for our presenters. If he sat down for a minute to rest his feet, he put on his other hat as Website Chair and checked to make sure the conference webpage was in working order.

Many of the new attendees were there as a result of **Marianne McFadden's** diligent work prior to the conference as Advertising Chair. Marianne was able to reach out to many teacher organizations and units across the state and provide materials informing them about the conference. **Kimberly Krout** and **Jake Maul** served as student volunteers and also clocked up many steps aiding the Conference Committee wherever they could. **Christian Good** communicated to conference participants as Social Media Chair and after learning of a presentation cancellation, put together a quick talk on using Social Media to fill the time slot.

I do hope you all completed the conference evaluation that was put together by **Nina Girard** and **Leslie Frischman**. The Conference Committee certainly appreciates all feedback as we continue to work to put together productive

FRINGE THOUGHT: Even happiness would lose it's meaning if it weren't balanced by sadness.

conferences to meet the needs of all math teachers across Pennsylvania. I would like to thank the management and conference staff of the Hilton Harrisburg for the wonderful experience they provided throughout the conference. They were available at all hours to see to the conference committee's needs and were positive and efficient in any last minute requests we had.

And finally, I want to extend my heartfelt gratitude to **Marian Avery** for her role as Conference Planning Committee Chair. Marian was our contact person with the Hilton and spent countless hours both on-site and from home working with the hotel staff to bring a conference setting that we look forward to returning to next year. No task was too difficult or too small for Marian and I want to thank her for her role in making this conference a success. Thank you!

Again, thanks to all of you who attended for coming. Many hours are spent planning and putting a conference like this together. It was very rewarding to hear so many positive comments from you throughout the conference. Please share what you learned at the conference. We will be at the same location next year August 6-7th 2018. Hope to see you all there.

Edel Reilly

2017 Conference General Chair

SAVE THE DATE…..AUGUST 6-7th, 2018 for the PCTM Annual Conference to be held at the Hilton Harrisburg, just a few blocks from the Pennsylvania Capitol.

Speaker proposal site to open in November with a submission deadline late winter.

Conference Registration opens April 1st, 2018.

More information to follow in the winter edition of the PCTM Magazine and on the PCTM website at www.pctm.org

FRINGE THOUGHT: The only sharp-edged tool that gets sharper with use is the tongue.

Pictures from the Conference

FRINGE THOUGHT: There's one good thing about living in the past. It's cheaper.

FRINGE THOUGHT: Lend an ear. And a chocolate.

PCTM Awards

Jo Kinsey

Thank you to all who nominated colleagues for PCTM awards. With your support we are able to recognize exemplary teachers who bring contagious enthusiasm and great skill to Pennsylvania students.

Hall of Fame Award

Steve Cicioni was recognized for his work as an educator with the Hall of Fame Award. Stephen has been the PCTM treasurer for 14 years and has expertly handled the finances of our organization and dealt with many changes in rules and tax laws governing a non-profit group. Steve continues to serve the teachers and students of the Commonwealth even in retirement. He works with Educational Mathematics Consultants providing professional development to teachers from Grade 3 through high school. His work with the Lancaster-Lebanon IU #13 has him providing classroom support as an instructional coach to middle and high school teachers throughout the school year. Since 2004, Steve has been an adjunct professor in the University of Pennsylvania's Graduate School of Education.

From 1993 to 2009 he taught in the Manheim Township School District. Prior to this, he taught mathematics to Grades 7-9 at Reynolds Junior High in the School District of Lancaster from 1972-1993. During the summer of 1979, he taught remedial math to incoming freshmen at the Stevens State School of Technology. Since 1984 Steve has served as a supervisor and teacher for the Tels Math Program. He has served as a consultant and facilitated numerous workshops for K-12 mathematics teachers. The workshop topics have covered the use of calculators as teaching and learning tools, the use of manipulatives to enhance instruction, problem solving, topics associated with the PSSA, and other areas aligned to the NCTM Standards. In addition, our treasurer presented workshops at conferences for both the Pennsylvania and the National Council of Teachers of Mathematics. The workshops addressed utilizing journal writing and investigative reporting as instructional strategies to enhance active learning. Steve served as a Facilitator for the PBS Mathline Middle School Math Project Learning Community, through station WITF in Harrisburg from 1997-1998.

In 1988, he co-authored the video disk "Interactive Mathematics." In 1990, as a Consultant for Ferranti Educational Systems, Steve worked as the Editor for Mathematics content of "Interactive Mathematics II", a video disk on algebra. Both disks were designed to train/remediate adults in college, business, and industry; both were marketed nationally. He co-authored - "An Action-Research Study on the Liberal Use of Calculators in a 7th Grade Mathematics Class" for the National Council of Supervisors of Mathematics Newsletter, October, Vol. XIX No. 1 in 1989.

Outstanding Contribution to Mathematics Education

Elizabeth "Lisa" Allen teaches math at Saucon Valley and is also as a part-time professor at Lehigh-Carbon Community College (LCCC). Lisa currently teaches AP calculus at Saucon Valley and also tutors calculus students after school. Her students participate in lessons that are dynamic, interactive, technology-based and problem-focused, utilizing a team approach for collaborative learning. Lisa is a past recipient of the Star teacher Award for Teaching with Technology. She has presented at conferences for the National Science Foundation, Lehigh University's Technology Summit, and the 2016 Pennsylvania Council of Teachers of Mathematics annual conference. She is the Pennsylvania recipient of The Presidential Award for Excellence in Mathematics and Science

FRINGE THOUGHT: Share something offline.

Teaching (PAEMST.) The PAEMST is awarded to outstanding K-12 science and mathematics teachers from across the country through the National Science Foundation. She also organizes an annual mathematics tour of New York City and its Museum of Mathematics. Lisa sits on the PCTM board as a member-at-large.

New Teacher Award

The New Teacher Award was created to recognize those who have been teaching for three years or less. Kate Hallinger has been teaching 4th grade at Radio Park Elementary School, State College for two years. When describing her classroom, Principal Zach Wynkoop tells us, "her classroom is a stimulating and vibrant place in which student engagement is the norm." Kate has worked with the instructional coach to create, implement, and assess a program which addresses the 8 mathematical practices and provides differentiation for her students. She recruited James Nolan, a retiree from State College Area School District as a classroom volunteer to support her lower ability students. James cited her use of the enquiry approach, math talks, and a continual search for new tools and strategies to reach her students and help them develop conceptual and procedural understandings in math.

Masters of Mathematics Award

The Masters of Mathematics Awards are given to educators who have taught 7 or more years in a school.

Intermediate Level

Stacey Miller has taught 4th grade at Pleasant Gap Elementary School in the Bellefonte Area School District for 15 years. Pleasant Gap principal Daniel (Duffy) Besch was on hand to present her award.

Stacey has gone from the student teacher who had anxiety about teaching math to the teacher you would go to as a resource for teaching. Her nominators all said she was innovative, analytical, resourceful, and constantly challenging herself. She integrates math into other subjects. Stacey has created units for science and math in collaboration with other teachers. She has been a mentor to student teachers. Their experience with Stacey is more like a co-teaching experience where they learn something new, design a program, implement it, differentiate instruction, and create assessments, which align with standards and curriculum. Outside of school hours, Stacey coaches high school soccer. She is also instrumental in planning and hosting Math competitions. Stacey has organized tutoring sessions before school 2 days a week and started an intervention/ enrichment group in her classroom. Stacey goes that extra mile to find how to best teach her students.

High School Level

John Farrington has taught mathematics at Archbishop Ryan High School in Philadelphia for 25 years. He teaches Honors Algebra II to juniors, Academic Level Algebra IB to sophomores, and College Prep Level Algebra I to freshmen. PCTM member and colleague Victoria Gradel received the award for Master of Math for John as he was unable to receive his award in person.

John is always looking for ways to incorporate real world problems into his curriculum. His most recent project revolved around a cross country trip which integrated measurement, area, slope, graphing, and financing for the trip. His students work cooperatively to solve problems. John has been an early proponent of technology use in the classroom. He not only helps students improve their use of technology, but will often take prep, lunch, and after school time to help

FRINGE THOUGHT: Ride with the top down.

colleagues incorporate technology into their teaching. He has also created programs for the administration to simplify their tasks. John has a tremendous sense of humor which he employs to hook his students on math. In addition to his classroom teaching, John has served as the girls' basketball coach and student council moderator. His mentoring has included numerous student teachers.

Past President

At the August conference, Marian Avery was recognized for her tenure as PCTM President. Marian has served PCTM in many capacities – Award co-chair, Secretary , and for the past 4 years as President-elect, President, and Past President. During that time, PCTM has seen many changes. Our constitution continues to undergo revision as we strive to meet the demands of the 21st century. Marian brought many new faces to the board from various regions of our commonwealth. She has also sought to increase membership in PCTM. Marian has strengthened our bond with NCTM. She has taken much personal time to attend conferences and makes PCTM's presence known as a group of educators of math. Our conference is now in the summertime. Marian is a vital force with organizing and dealing with the minutiae of conferences. We know she will continue to do so and will always be involved with PCTM. We are appreciative of her many efforts.

New Attendee Scholarship

Each year we encourage a math educator who has never been to a conference to join us and learn from other teachers. Tekla Hilton was this year's awardee of the scholarship.

FRINGE THOUGHT: Accept a compliment.

Raffle Winners from the 66th Annual PCTM Conference evaluation survey are:

$25 NCTM gift certificate

Carie Michael from Oxford School District in Chester County

Free PCTM Conference registration to next year's conference at Harrisburg Hilton:

Daniel Kaufman from Wilson School District in Berks County

$25 Best Buy gift card (donated by Algebra by Hand):

Dave Kennedy from Shippensburg University

Attention Classroom Teachers and Student Affiliates

The Mathematics Education Trust (MET) of the National Council of Teacher of Mathematics (NCTM) channels the generosity of contributors through the creation and funding of grants, awards, honors, and other projects to support the improvement of mathematics teaching and learning.

Apply for NCTM's Mathematics Education Trust grants, scholarships, and awards. Funding ranges from $1,500 to $24,000 and is available to help math teachers, prospective teachers, and other math educators improve the teaching and learning of mathematics.

For more information about the various grants, scholarships, and awards go to http://www.nctm.org/Grants/

Didax Heinemann Dedicated to Teachers ETA hand2mind Nasco EAI Education

Pre-Service Teacher (PST) Day

Saturday, October 21, 2017
9:30am-3:30pm

Exchange teaching ideas, experiences, and practices!

Kick start your professional mathematics teaching career!

Network with other pre-service teachers from across the state!

Select **one** of two locations!

Eastern PA: Millersville University of PA

- As part of the Careers in Mathematics Conference at Millersville University—sponsored by EPaDel, PCTM, & PAMTE
- Registration: $25
  - Registration link: http://sections.maa.org/epadel/meetings/2017/careers/
  - Deadline: Oct. 7th, late registration $30

Session Topics for PreK-12 PST include:

- Guided Math for Elementary
- STEM Education for 7-8th
- Educational Technology Apps for Middle Level Math
- Population Education
- Math Software
- Math Design Collaborative materials
- Number Talks
- Interactive Notebooks and Foldables

Western PA: Indiana University of PA

- Held in conjunction with the 5th Annual LHMA Mini-Conference—sponsored by LHMA, PCTM, & PAMTE
- Registration: $25 (PSTs) & $30 (Others)
  - Registration link: https://docs.google.com/forms/d/e/1FAIpQLSe7yr2b8K7WZyhbkhHkJy61MXQOZAfpRTvq70R2BgMAUYyrrA/viewform
  - Deadline: Oct. 7th, no onsite registration

Opening remarks by current/ former PCTM Presidents.

For more information contact:
Eastern PA: Cynthia Taylor - cynthia.taylor@millersville.edu OR 717.871.7317
Western PA: Valerie Long – vlong@iup.edu OR 724.357.4060

Laurel Highlands Mathematics Alliance pAmte PCTM Eastern Pennsylvania EPaDel Delaware

FRINGE THOUGHT: Hit snooze. Again.

2017 PCTM Election Results

PCTM President-Elect - Kelly Brent
Delegate at Large - Jennifer Umberhocker
Western Regional Representative - Becky Piscitella
Secretary - Courtney Nagel
Eastern Regional Representative - Dave Frederickson

President-Elect

Kelly Brent is the Mathematics Department Chair for Carlisle Area School District, grades 6-12. She teaches the most gifted mathematics students in AP Calculus BC and the most struggling students in Integrated Math III in the morning and attends to her departmental responsibilities in the afternoon. Kelly is passionate about mathematics and says she feels blessed to teach this range of students who keep her on her toes in different ways. Kelly is an adjunct professor at Shippensburg University and Harrisburg Area Community College where she teaches a wide range of courses from developmental mathematics to Calculus. She has served on the PCTM Board for the past five years in various roles including Delegate at Large, Membership Growth, and 2015 Conference Co-Chair. Her enjoyment in sharing ideas with colleagues across the state on how to prepare students for college-level mathematics, state assessments, and career/life oriented problem solving has inspired her to present workshops on formative assessment and preparing students for the Keystone Exam. Kelly says, "I have enjoyed sharing my ideas and attending workshops gaining new exciting strategies at every PCTM conference. Being a part of PCTM has enriched my professional life in so many ways. I look forward to the opportunity to continue to be part of this organization."

Secretary

Courtney Nagle is an Assistant Professor of Mathematics Education and Program Chair of the Secondary Mathematics Education Program at Penn State Erie, The Behrend College. Courtney teaches both math content courses and math methods courses at Penn State Erie, and she is actively engaged in research. She has served as the Principal Investigator for two funded National Science Foundation grants. In addition, she established an annual Best Practices in Teaching Mathematics conference held at Penn State Behrend each summer. Recent accolades include being named the University at Buffalo Department of Learning and Instruction Outstanding Alumni and receiving the 2015 Penn State Behrend Council of Fellows Excellence in Outreach Award.

Delegate at Large

Jennifer Umberhocker earned her Masters Degree in Education from Cabrini University outside of Philadelphia and currently lives in Lancaster, PA. She began teaching math to middle school students at Linden Hall in 2008. Throughout her time there, she also taught ESL and served as Dean of Students. Jennifer worked with the math department to prepare students for the AMC 8 and AMC 10 math tests, as well as coached the middle school MathCounts team. Her passion for teaching young people to not be afraid of math, to see it as a series of steps to solve rather than one big problem, led her to her most recent work, Assistant Head of School at the Montessori Academy of Lancaster where she works with young students, grades 1-6, as well as tutoring local students in math and other subjects. Jennifer also spends time volunteering with CASA and serves on the board of Children

FRINGE THOUGHT: When people tell you he got rich through hard work, ask them whose?

Deserve A Chance, a non-profit organization that "provides students the opportunity to learn about the college application process, develop professional and personal skills, and work with mentors and tutors to maintain an exceptional educational status."

Easter Regional Representative

Dave Frederickson has been a member of PCTM since July 2015. He is pursuing a M.S. degree in Mathematics Education from Northern Arizona University distance learning program with an expected graduation date of Summer 2019. He is the 8th grade math department chair in his fifth year of teaching at Commonwealth Charter Academy, a public charter cyber school with locations throughout PA – his location where he works is Andreas, PA near Lehighton. Prior to this he worked 2 years (full-time) at KidsPeace, a Children's Psychiatric Hospital in Schnecksville, PA. He taught in the School District of Philadelphia where he was part of Cohort 7 for the Philadelphia Teaching Fellows program, 2009-2010. "My objective for the remainder of my life is to reach and inspire as many students as I can. Not until recently, did I expand this perspective and place a huge emphasis on professional development for colleagues and myself alike. As colleagues and I grow, then students grow, and we all grow as a result."

Western Regional Representative

Becky L. Piscitella, D.Ed. has been teaching high school mathematics for 18 years at Richland High School in Johnstown, PA. In 2009, she became the Pennsylvania recipient of the Presidential Award for Excellence in Mathematics and Science Teaching, the highest award a math teacher can receive in the United States. Dr. Piscitella attended the University of Pittsburgh at Johnstown to complete a B.S. in Secondary Education Mathematics, a B.S. in Applied Mathematics, and a Minor in Computer Science. She later attended Indiana University of Pennsylvania, Indiana, PA to earn a M.Ed. in Mathematics Education. This past December, Dr. Piscitella completed her research and successfully defended her dissertation at Indiana University of Pennsylvania to earn a D.Ed. in Curriculum and Instruction. In addition to the Presidential award, Dr. Piscitella was featured in a Pitt-Johnstown Magazine STEM article and she received Congressional recognition in June of 2010. Over the years, she has mentored 6 student teachers, presented several times at PCTM, served as PCTM Western Regional Representative this past year, serves as the webmaster for the Laurel Highlands Math Alliance, and has published articles and a book chapter. Dr. Piscitella also served as the committee chair, co-editor, and co-author of the 2012 edition of Hershey Park Education Guides Math Lab Manual. She donates time to Penn Highlands Community College on the Advisory Committee for Education Programs and reviews articles for *Early Childhood Education Journal.* In her free time, Dr. Piscitella enjoys being involved in her 3 children's education and activities. In the evening, she and her husband can be found relaxing on the couch with their lab and golden retriever, Louis & Clark.

FRINGE THOUGHT: What makes life worth living is giving and forgiving.

Access to Member's Section of PCTM Website

To login to the members only section:

Go to: www.pctm.org

click on members only

username: **member**

password: **mF7eoG**

2017 Pennsylvania Statistics Poster Competition

Pete Skoner

The Science Outreach Center (SOC) at Saint Francis University is pleased to announce the winning posters for the 2017 Pennsylvania Statistics Poster Competition, hosted for the ninth year by Saint Francis University. The annual state competition is in its twenty-first year overall, and is coordinated by the SOC and the Department of Mathematics, Engineering and Computer Science. Cash awards of $96 for first place, $72 for second, $48 for third, and $24 for fourth in each of four grade levels are awarded to the students who submitted winning posters. External financial support is provided by several regional and state professional organizations, including the Pennsylvania Council of Teachers of Mathematics, Association of Teachers of Mathematics of Philadelphia and Vicinity, Laurel Highlands Mathematics Alliance, Mathematics Council of Western Pennsylvania, and the Harrisburg, Philadelphia, and Pittsburgh Chapters of the American Statistical Association (ASA). A statistics poster is a display containing two or more related graphics that summarize a set of data, look at the data from different points of view, and answer specific questions about the data. The 564 posters submitted included 111 in the K-3 grade category, 216 from grades 4-6, 120 from grades 7-9, and 117 from grades 10-12. These 564 posters were submitted from a total of 1,026 students! Judging for the state competition was held on Saturday, March 11, 2017. Winning posters were then submitted to the National Statistics Poster Competition, which is coordinated by the American Statistical Association. One of the Pennsylvania winning posters in the K-3 category was awarded 1st place in the National competition; congratulations to student Marguerite Burstym and teacher Bonnie Howe from the Highland Elementary School in the Abington School District for the poster *Chocolate Melting Time*. And congratulations to all the students who created and submitted the winning posters, and to the teachers who guided the students.

Grades K-3 Winning Posters 2017

First Place: Rowan Cooper, *Supported or Unsupported,* East Pike Elementary School, Taught by Elizabeth Woods
Second Place: Sophia Burgos, *How Many Grown-Ups Can Pass A Third Grade Science Test?,* Roslyn Elementary School, Taught by Tyheshia Carmichael
Third Place: Marguerite Burstyn, *Chocolate Melting Time,* Highland Elementary School, Taught by Bonnie Howe; * First Place in the National Statistics Poster Competition *
Fourth Place: Joshua Reed, *The Oreo Project Oreo Cookies vs. Milk,* East Pike Elementary School, Taught by Elizabeth Woods

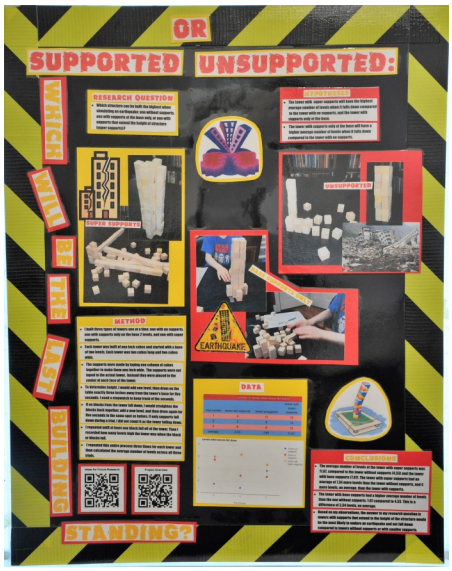


Figure 1. Grades K-3 1st Place poster by Rowan Cooper.

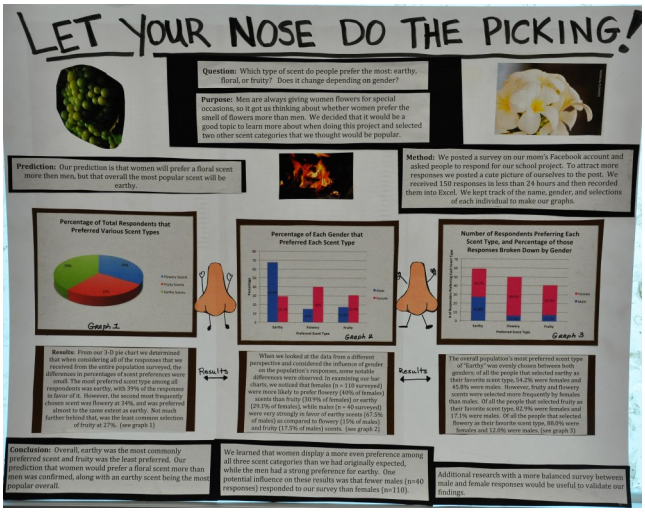


Figure 2. Grades 4-6 1st Place poster by Oscar Hoffmann and Calvin Hoffmann.

FRINGE THOUGHT: People who wrestle with their consciences usually go for two falls out of three.

Grades 4-6 Winning Posters 2017

First Place: Oscar Hoffmann and Calvin Hoffmann, *Let your nose do the picking?,* Rydal Elementary School, Taught by Pamela DaSilva
Second Place: Giovanni Corabi, *How Many Minutes of an American Football Games are Live Action?,* Rydal Elementary School, Taught by Pamela DaSilva
Third Place: Max Farma, *Ouch! Youth Sport Injuries,* Rydal Elementary School, Taught by Sarah Finnegan
Fourth Place: Claire Piper, *Go For the Gold,* Baker Elementary School, Taught by Lori Piper

Grades 7-9 Winning Posters 2017

First Place: Emily Wang and Joy Liu, *The Illusion of Superiority,* State College Area High School, Taught by Eric Davy
Second Place: Gus Robinson, *Which airfoil design will create the most lift?*, Campus School of Carlow University, Taught by Suzanne Ament
Third Place: Evan Blair, *Which Soda has the Worst Effect on Teeth?,* Campus School of Carlow University, Taught by Suzanne Ament
Fourth Place: John Schroeder, *Head Protection*, Campus School of Carlow University, Taught by Suzanne Ament

Grades 10-12 Winning Posters 2017

First Place: Lauren Betar, Kaelie Goss, Chris Phillippe and Carly Ebersole, *Does a Parent-Teacher Conference Help Students?,* Bishop Guilfoyle Catholic High School, Taught by Sue Provenzano
Second Place: Lauryn Zelnosky, Natalie Khoury, Marguerite Nosek and Alana Hallinan, *Old School vs. New School,* Bishop Guilfoyle Catholic High School, Taught by Sue Provenzano
Third Place: Sydney Bules, Lexi Conner, Lauren Garrity and Linda Zamrowski, *Saving Species, Saving Lives,* Garnet Valley High School, Taught by Rudolph Shiller
Fourth Place: Ruth Deangelo, Casey Snyder and Rosa Padt, *Peeking Without Permission,* State College Area High School, Taught by Eric Davy

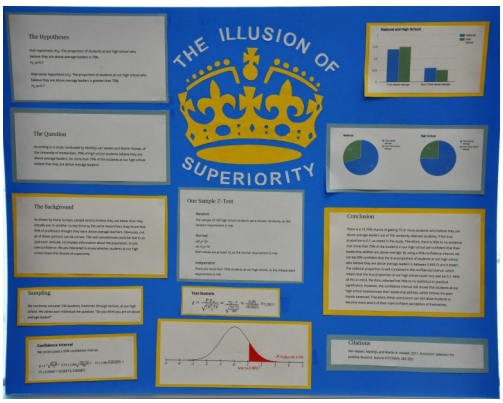


Figure 3. Grades 7-9 1st Place poster by Emily Wang and Joy Liu.

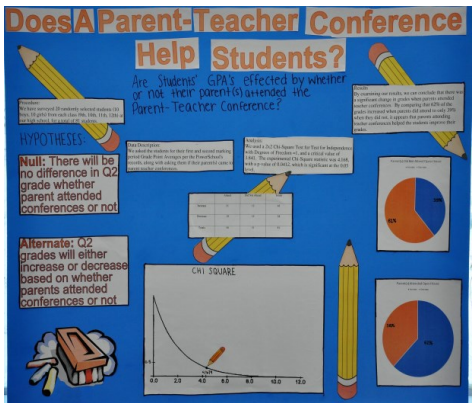
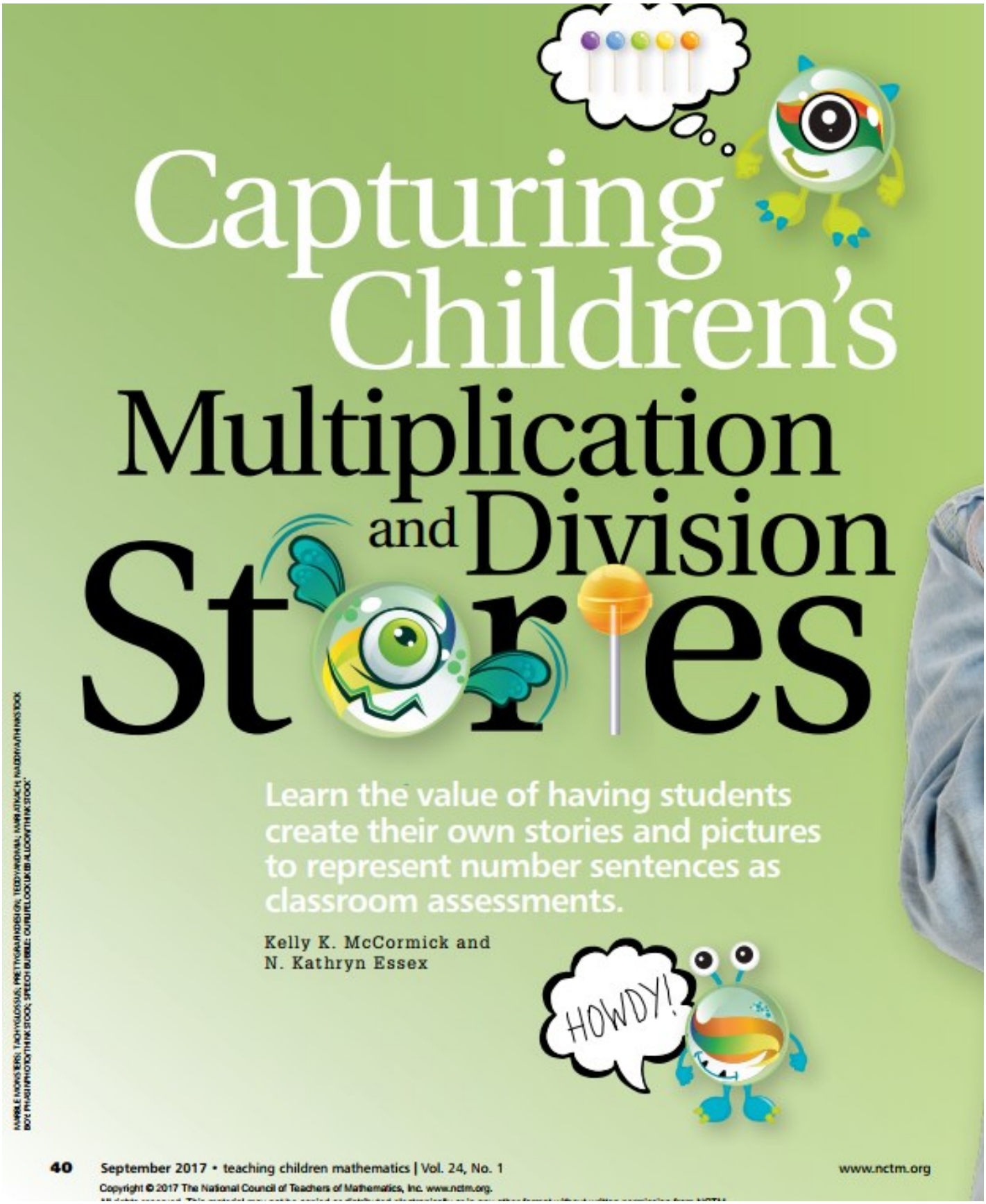


Figure 4. Grades 10-12 1st Place poster by Lauren Betar, Kaelie Goss, Chris Phillippe, and Carly Ebersole.

Dr. Peter Skoner is a Professor of Physics at Saint Francis University in Loretto, Pennsylvania. He coordinates the Pennsylvania Statistics Poster Competition at Saint Francis University.

FRINGE THOUGHT: We flatter those we scarcely know and please our fleeting guest, but render many a heartless blow to those we love the best.



Capturing Children's Multiplication and Division Stories

Learn the value of having students create their own stories and pictures to represent number sentences as classroom assessments.

Kelly K. McCormick and
N. Kathryn Essex

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A third-grade student wrote the following repeated-addition story as part of an assessment given to 583 third graders near the end of the school year (see **fig. 1**).

> Once a group of five little marbles were walking. They ran into five more marbles. Now the five is ten. Five more marbles came by. Now a group of fifteen marbles are walking, and then they all bought lollipops.

We had asked the children to "make up a story and a picture about marbles for this number sentence: $3 \times 5 = 15$." Students in this study came from predominantly low- to average-income families living in three distinct geographical areas within the United States. We also collected work, which included a similar division task, from these students at the end of their fourth-grade year. In this article, we present findings describing the children's multiplication and division stories and discuss the value of having students create their own stories and pictures as classroom assessments.

We wanted to capture and examine the children's understanding of multiplication and division. Research suggests that providing a foundational understanding of the meaning of an operation supports students' competence in problem solving and computation (Fuson 2003). Correspondingly, understanding multiplication is a powerful tool; multiplication is a primary operation that can be properly defined so it is fundamental for representing and solving many different situations (Otto et al. 2011). When solving word problems, children—

> frequently choose an operation without making sense of the choice. . . . Knowing why an operation is an appropriate choice for a solution strategy is an important part of establishing a robust understanding of mathematics. (Otto et al. 2011, p. 15)

FRINGE THOUGHT: The purpose of education is to replace an empty mind with an open one.

FIGURE 1

As part of an end-of-the-year assessment, a third grader wrote this repeated-addition story and drew a picture for the number sentence 3 × 5 = 15.

Story:

Once a group of 5 little marbles were walking. they ran into 5 more marbles. Now the 5 is 10. 5 more marbles came by. Now a group of 15 marbles are walking, and then they all bought lolly-Pops.

Picture of marbles:

FIGURE 2

This child wrote a story in which the mathematical structure was 5 + 5 + 5 = 15.

Story:

Tom has 5 mabils he fond 5 more. Then he fond 5 more how mony does he have

Picture of marbles:

FIGURE 3

Here is an example of a third-grade student's picture and multiplicative story about equal groups for the number sentence 3 × 5 = 15.

Story:

Lucy had 3 jars of marbles. Each jar had 5 marbles in it. How many marbles did Lucy have?

Picture of marbles:

The Common Core State Standards for Mathematics (CCSSM) (CCSSI 2010) clearly emphasizes the importance of understanding the meaning of multiplication and division; CCSSM states that developing an understanding of multiplication and division is one of four critical areas in third grade, when students are to—

> develop an understanding of the meanings of multiplication and division of whole numbers through activities and problems involving equal-sized groups, arrays, and area models; this includes understanding the meanings of whole number multiplication and division. (CCSSI 2010, p. 21).

Third graders are to—

> interpret products of whole numbers, e.g., interpret 5 × 7 as the total number of objects in 5 groups of 7 objects each. For example, describe a context in which a total number of objects can be expressed as 5 × 7. (CCSSI 2010, p. 23)

These standards state that third graders should be able to—

> interpret whole-number quotients of whole numbers, e.g., interpret 56 ÷ 8 as the number of objects in each share when 56 objects are partitioned equally into 8 shares, or as a number of shares when 56 objects are partitioned into equal shares of 8 objects each. For example, describe a context in which a number of shares or a number of groups can be expressed as 56 ÷ 8. (p. 23)

CCSSM extends this focus to fourth grade, when students should be able to "interpret a multiplication equation as a comparison, e.g., interpret 35 = 5 × 7 as a statement that 35 is 5 times as many as 7 and 7 times as many as 5" (p. 29) and to the fifth, sixth, and seventh grades, when students should learn to "apply and extend previous understandings of multiplication and division" to fractions and rational numbers" (pp. 34, 41, and 48).

Having children create their own multiplication and division stories and pictures provided us with rich information about their understanding of these operations. As educators, we could show students how to represent a problem situation, but we learn more about their understanding of the operation and quantities involved when they create their own stories (Otto et al. 2011). In addition, because we recognize the importance of children concurrently developing an understanding of multiplication and division and the relationship between the operations (CCSSI 2010; Fosnot and Dolk 2001; Mulligan and Mitchelmore 1997; Otto et al.

2011; Russell 2010; Van De Walle et al. 2013), we collected work that represented the children's understanding of both operations.

Children's multiplication stories

To better examine children's understanding, we designed tasks that allowed us to explore the different types of multiplication and division stories that children compose. How children model a situation reflects their reasoning, and their explanation of their thinking guides their way of representing the situation and any expression or equation that they write (Otto et al. 2011). Thus, we decided to ask the children to create narratives and diagrams about marbles to model equations. Being provided with the context of marbles allowed the children to focus on the mathematics in the task; it made the task less time intensive by narrowing the possibilities of the stories' context and by reducing the time it took for children to draw a diagram, while still capturing their understandings of the meanings of multiplication and division. We then developed a coding scheme based on their work. We discovered that the children's correct multiplication stories could be categorized into the following groups: stories—

- about repeated addition;
- about equal groups;
- with both multiplicative and additive aspects present; and
- about comparison situations.

We coded stories about adding three things five times or five things three times and multiplication stories about repeated addition. With these types of stories, children wrote about having one group of five marbles, then having another, and then a third. For example, in **figure 2**, the child writes a story for which the mathematical structure is $5 + 5 + 5 = 15$.

The multiplicative stories were about equal groups (see **fig. 3**): either three groups of five things or five groups of three things. The development from repeated addition to multiplication requires children to understand a higher-order treatment of number, unitizing, in which groups are counted as well as the objects in the group (Fosnot and Dolk 2001). Children must be able to think about the numbers involved with groups of objects in two ways: each group represents one thing at the same time it is a number of things. Before constructing the idea of unitizing, number is used to represent single units—six represents six marbles (Fosnot and Dolk 2001).

As Fosnot and Dolk (2001) note, children do not construct mathematical ideas in any set or ordered sequence. "They go off in many directions as they explore, struggle to understand, and make sense of the world mathematically" (p. 18). We saw evidence of this when some of the children's stories contained elements of both additive and multiplicative thinking. For example, the story in **figure 4** starts with one group of marbles, and then two more groups are added.

Last, we also found multiplicative comparison stories. Multiplicative-compare situations are about two sets; one set is a multiple of the other (Van De Walle et al. 2013). In these stories, a comparison is made between the amount in one group and the amount in a number of groups of the same size. For example, in the story shown in **figure 5**, the number of marbles in one group of three is compared to the number in five groups of three.

Of the 583 students who were a part of this

FIGURE 4

A third-grade student's story has both multiplicative and additive aspects present as well as a picture for the number sentence $3 \times 5 = 15$.

Story: Mark had a bag of 5 marbles. He found 2 more bags of 5 marbles now he has 15 marbles.

Picture of marbles:

FIGURE 5

For the number sentence $3 \times 5 = 15$, one third grader wrote a multiplicative-compare story and picture.

Story: I have 3 marbles. My friend has 5 times of that amount. How many marbles does my friend have?

Picture of marbles:

MARBLE MONSTERS: TAOHYOLOBUS/THINKSTOCK; PRETTYGRAPHIC DESIGN/THINKSTOCK; TEDDYWINDMA/THINKSTOCK

FRINGE THOUGHT: No one wants to steal your troubles. No one can steal your good deeds.

FIGURE 6

This child's multiplicative story and picture for the number sentence 3 × 5 = 15 is incorrect.

Story: Sharlit cho-cho had 3 marbuls mark gave her 3 more she wanters if he had given her 5-3 how many it would equil help her

Picture of marbles:

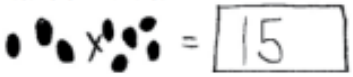
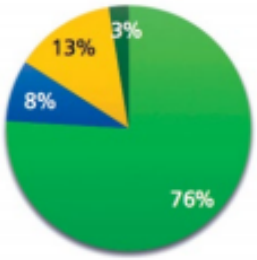


FIGURE 7

The majority of students who wrote correct stories had constructed the big idea of unitizing, which underlies the understanding of place value, multiplication, and division.

Third graders' correct multiplication stories



Multiplicative	262
Additive	28
Additive and multiplicative	46
Multiplicative-compare	9

study, 345, or 59 percent, wrote a correct story; 312 children also drew a correct picture. Fifty-two children wrote an incorrect story but drew a correct picture. **Figure 6** shows an example of an incorrect multiplication story and picture. Among the 345 correct stories, 262 were multiplicative stories, 28 were additive stories, 46 stories had both additive and multiplicative aspects, and 9 were comparison stories (see **fig. 7**). Thus, the majority of students who wrote correct stories exhibited the ability to think about equal groups; they had constructed the big idea of unitizing, which underlies the understanding of place value, multiplication, and division (Fosnot and Dolk 2001).

Children's division stories

The fourth-grade division story task mirrored the multiplication task:

> Make up a story and a picture about marbles for this number sentence: 18 ÷ 6 = 3.

Once again, we looked at not only the correctness of students' responses but also the different types of stories composed. Division stories, like multiplication stories, are about equal-size groups.

We coded stories about sharing items equally across a given number of groups as fair-sharing or partitive-division stories. The three examples of the fourth graders' stories (see **fig. 8**) demonstrate that these children have a strong understanding that 18 ÷ 6 = 3 means that eighteen objects are divided equally into six groups and that there are three objects in each of the groups. Their stories show they understand that the groups must be equal, and their questions focus on the number of objects in each group.

Repeated subtraction or measurement division stories had contexts in which a given number of marbles were repeatedly subtracted from the whole group. We found fewer examples of these stories in fourth graders' work. Three examples of fourth graders' stories (see **fig. 9**) demonstrate that these students have a strong understanding that 18 ÷ 6 = 3 means that eighteen objects are divided into groups of six and that three groups of six objects are in eighteen. With a repeated subtraction problem (see **fig. 9a**), the question is, "How many sixes are in eighteen?" However, because we had

asked the children to "make up a story [not a problem] and a picture about marbles for this number sentence: 18 ÷ 6 = 3," the story shown in **figure 9b** also demonstrates a correct understanding of division. Even though the child does not directly pose the question, "How many sixes are in eighteen?" she explains how to determine the number of sixes that are in eighteen by repeatedly subtracting six from eighteen, twelve, and then six; and as the child then states, after that, "there were no marbles left, so the answer was 3." If this were a classroom assessment, a follow-up question would be to ask this student what the three in her story means to ensure that she understands that the three is three groups of six.

Some children wrote correct stories for this task that we coded as multiplication stories (see **fig. 10**). In other words, the action in the story was about finding out how many marbles were in three groups of six objects or six groups of three. This suggested to us that these children understood that multiplication and division may be used to represent the same situation, that is, situations involving a given number of equal-size groups. Mulligan and Mitchelmore (1997) also found that children naturally relate these two operations and that when they do, they do not necessarily find one more difficult than the other, again emphasizing the importance of providing children with opportunities to link the operations of multiplication and division more closely.

In the fourth grade, 356, or 61 percent of the 583 students, wrote a correct division story, and 309, or 53 percent of all the children, also drew a correct picture. An additional 42 students drew a correct picture but did not write a correct story. Among the correct stories, 280, or 79 percent, were stories with fair-sharing contexts. Another 46 students, or 13 percent, wrote stories with repeated-subtraction contexts. Eleven students wrote equal-group division stories that had neither a fair-sharing nor a repeated-subtraction context. One child wrote a comparison story. Eighteen students wrote multiplication stories.

In the classroom

Research highlights students' difficulty solving story problems; they often guess at which operation to use to solve a problem if they do not understand what the operations mean (Verschaffel et al. 2007). For students to develop adaptive expertise in interpreting problems and carrying out appropriate computation to solve them, instruction—including assessment—must emphasize students' understanding of the action and the meanings of the operations in context (Russell 2010). Understanding the multiple meanings of operations and the relationship between the meanings and operations is a critical part of establishing

FIGURE 8

Three students' fair-sharing stories and pictures show that these fourth graders have a strong understanding that 18 ÷ 6 = 3 means that eighteen objects are divided equally into six groups with three objects in each.

(a)

Story: Me and my 5 friends have 18 marbles We are trying to make it ecle. How many can we all have?

Picture of marbles:

(b)

Story: Fred has 18 marbles and he wants to split them between his six friends, Charlie, Bob, Frank, Chuck, Hank, and Meatloaf. If he split his marbles evenly into six group. how many marbles would his friends each have?

Picture of marbles:

(c)

Story: six robber plan to steel 18 marbels from a toy shop. If they steel all 18 and deride them evidly between them robber get? marbles would each

Picture of marbles:

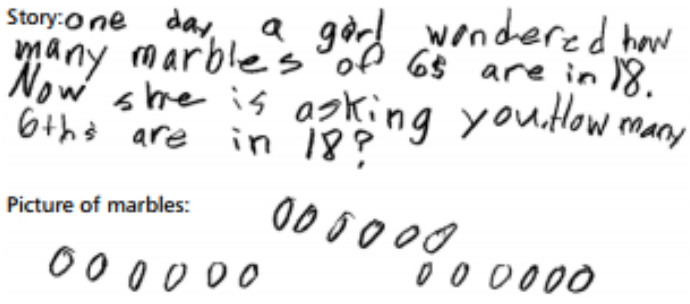
MARBLE MONSTERS: TACHYOLOBUS/THINKSTOCK; PRETTYGRAPHICDESIGN/THINKSTOCK; TEDDYVADIM/ATHINKSTOCK

FRINGE THOUGHT: Time you enjoy wasting is not wasted time.

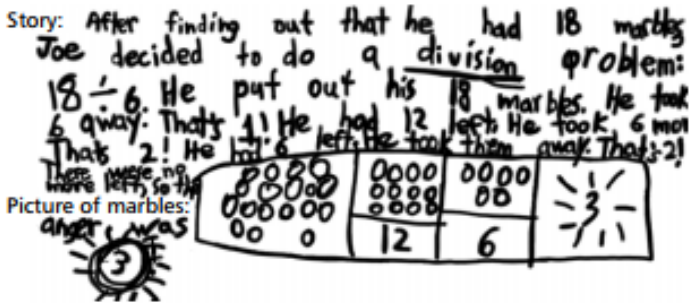
FIGURE 9

Three fourth graders' stories demonstrate that these students have a strong understanding that 18 ÷ 6 = 3 means that eighteen objects are divided into groups of six and that three groups of six objects are in eighteen.

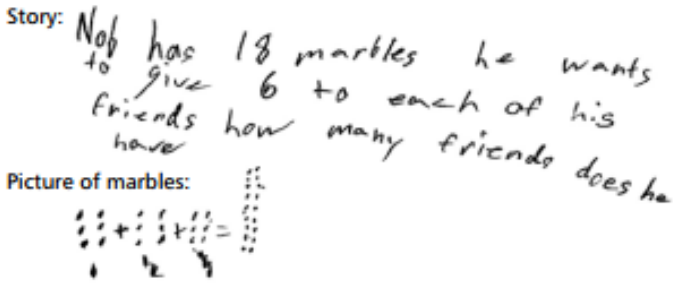
(a) With a repeated subtraction problem, the question is, "How many sixes are in eighteen?"



(b) Although this student never directly posed the problem, her example shows a correct understanding of division.



(c)



a strong, foundational understanding of mathematics (Otto et al. 2011).

Having students create their own stories and pictures to represent number sentences gave us a snapshot of their understanding and ability to apply the meaning of the operations to the context of marbles. Likewise, having children generate stories or story problems for a given equation or expression is a powerful way for classroom teachers to assess students' knowledge of the action and meaning of the operations (Drake and Barlow 2007; Van De Walle et al. 2013). It requires a higher level of thinking than simply solving a variety of story problems, which is how teachers typically assess students' understanding of the meaning of the operations.

The stories and diagrams that children create offer a multitude of opportunities for teachers to facilitate rich classrooms discussions about the different meanings of the operations and how the meanings are related. For example, a teacher might frame a discussion around having students compare a child's repeated-addition multiplication story to another child's equal-groups multiplication story. Similarly, another discussion could be built around having students compare a child's fair-sharing division story to another child's repeated-subtraction problem. When exploring part-whole relationships, asking children, "What did you know?" and "What were you trying to find out in your problem?" is a powerful tool (Fosnot and Dolk 2001). Through the previous discussion, students should come to realize that with both types of division stories, the total number of marbles is known. However, in one story, they are trying to figure out how many marbles are in each group; and in the other, they know how many marbles are in each group and are trying to figure out how many groups. Building on the previous discussion, having the children determine which interpretation of division is involved (how many groups or how many in each group) in other children's stories would deepen their operation sense. Other follow-up discussions might focus on the following questions: "Some of the how-many-groups stories say that the groups need to be equal or the marbles need to be shared evenly [*see* **fig. 8a, b,** *and* **c**]; is it important that the groups are equal and that the marbles were shared evenly? Why did you include that in your story?"

Using problem writing as an assessment reveals students' understandings and misunderstandings of operations in a manner in which traditional assessments cannot (Drake and Barlow 2007–2008). The most powerful part of the learning experience described previously is that the stories come from the children. As students write and discuss their problems, they

reveal their mathematical thinking; the value of student discussions, such as those previously described, is quite evident from the NCTM (2000) Standards and the Common Core's (2010) Standards for Mathematical Practice.

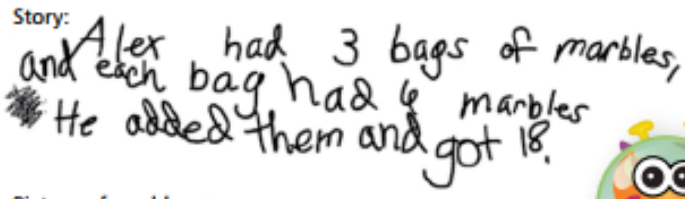
Common Core Connections

3.OA.1
3.OA.2
4.OA.1

FIGURE 10

Stories like this fourth grader's multiplication story and picture for the number sentence 18 ÷ 6 = 3 show that some children naturally relate multiplication and division and do not necessarily find one more difficult than the other.

Story:



Picture of marbles:

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Ed. note: For more on this topic, consult *Multiplication and Division in Grades 3–5* in NCTM's Essential Understanding series.

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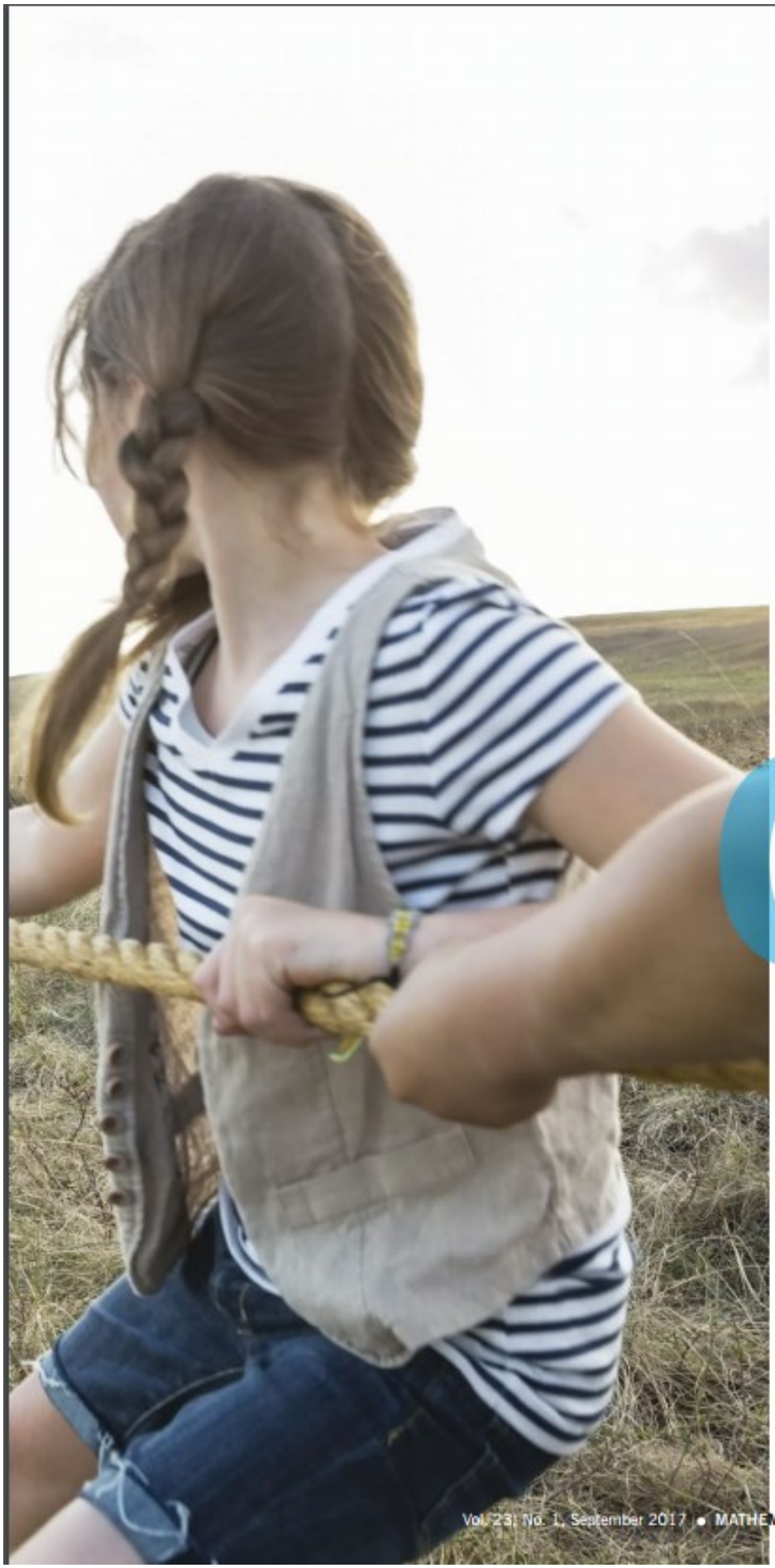
FRINGE THOUGHT: You can either complain because roses have thorns or you can rejoice because thorns have roses. –Ziggy

RETURN OF THE Tug-of-War

FRINGE THOUGHT: An old-timer is a man who's had a lot of interesting experiences—some of them true.



In this updated version of a Marilyn Burns task, students use algebraic reasoning to determine the outcome of a contest involving rebel fighters, storm troopers, and (of course) Chewbacca.

JULIE MCNAMARA

One of my favorite problems to share with students and teachers is Marilyn Burns's "Mathematical Tug-of-War" (Burns 2015) (see **fig. 1**). Long before the release of the Common Core State Standards (CCSSI 2010), the Mathematical Tug-of-War was engaging students in the type of reasoning and problem solving described by the Standards for Mathematical Practice (SMP).

I cannot remember when I was first introduced to the Mathematical Tug-of-War, but over the years I have shared it with many students and countless teachers. I have often used the problem in its original form, but I also like to rewrite it to reflect popular culture or current events. For example, some versions from the past included characters from *The Simpsons* or from the Harry Potter books.

FRINGE THOUGHT: If at first you don't succeed, think how many people you've made happy.

For my most recent version of tug-of-war, I used characters from the newest Star Wars movie, *The Force Awakens*. I knew that many of the sixth graders in the math class I was co-teaching once a week were fans of the franchise. (See the **activity sheet** on page 48.)

To prepare for the lesson, I looked for images to represent the characters in the tug-of-war. Instead of teams of acrobats and grannies (as in the original version of the problem), Round 1 involved five rebel fighters (Rey, Finn, Princess Leia, BB-8, and R2-D2) and four storm troopers, with each team being of equal strength. For Round 2, one team consisted of Rey, Finn, and one storm trooper. On the other side was Chewbacca, playing the role of Ivan in the original version of the task. Again, both sides were equal in strength. The final round, and the pull for which students needed to figure out the outcome, involved Chewbacca, Rey, Finn, and Princess Leia on one side and the four storm troopers on the other.

SETTING UP THE PROBLEM

To provide students with opportunities to make sense of the problem and motivate them to persevere in solving it (SMP 1), I first showed them a picture of a tug-of-war to make sure the context was familiar to them. I then asked for volunteers to demonstrate a "mathematical tug-of-war." Several hands shot up, and I had no problem getting enough volunteers to represent Round 1. As the volunteers came to the front of the room, I hung a picture of either a rebel fighter or a storm trooper around their necks and directed them to make two teams. I also gave them a short length of rope to further represent the tug-of-war.

At this point, we paused, and I asked all students, those involved in the tug-of-war and those still seated, to describe what they saw. Students agreed that Round 1 showed that the team of the five rebel fighters and the team of the four storm troopers were equal in strength (see **fig. 2**).

For Round 2, I called one last volunteer to the front of the room to represent Chewbacca. I then directed the students representing Rey, Finn, and one storm trooper to make one team and the student representing Chewbacca to make the other team (see **fig. 3**).

We again paused and described what we learned from Round 2. Students said things like "Chewy is really strong—as strong as three people!" From this comment, I knew that some students were equating the strength of all of the other characters, both rebel fighters and storm troopers, so I knew that this was something I would likely have to address as students worked on their solutions.

For the final round, I rearranged the students so that we had Chewbacca and three rebel fighters (Rey, Finn, and Princess Leia) on one side and the four storm troopers on the other (see **fig. 4**).

We once again took a good look at the two teams to make sure that all students understood what happened in Round 3.

Fig. 1 The original Mathematical Tug-of-War (Burns 2015) involved acrobats, grandmas, and a dog named Ivan.

A Mathematical Tug-of-War

Use the information given to figure out who will win the third round in a tug-of-war.

Round 1: On one side are four acrobats, each of equal strength. On the other side are five neighborhood grandmas, each of equal strength. The result is dead even.

Round 2: On one side is Ivan, a dog. Ivan is pitted against two of the grandmas and one acrobat. Again, it's a draw.

Round 3: Ivan and three of the grandmas are on one side and the four acrobats are on the other.

Who will win the third round?
Write an explanation of your reasoning.

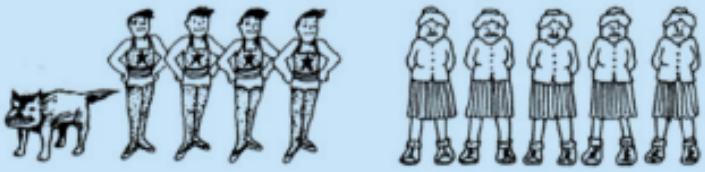


Fig. 2 In Round 1, five rebel fighters were equal to four storm troopers.

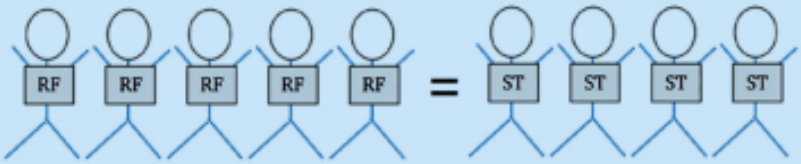


Fig. 3 In Round 2, two rebel fighters and one storm trooper were equal in strength to Chewbacca.

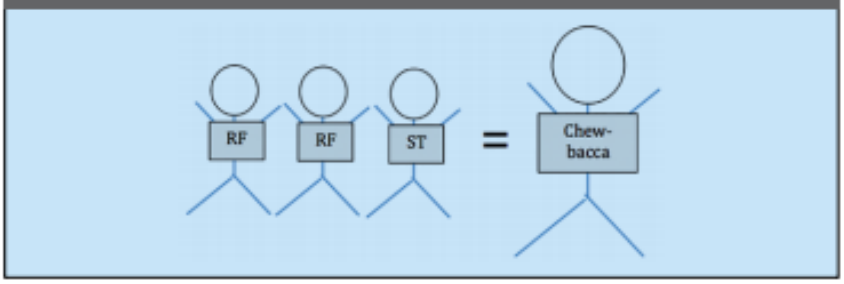
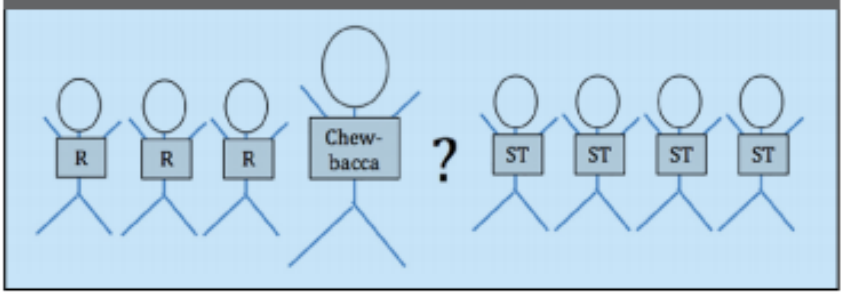


Fig. 4 In Round 3, three rebel fighters and Chewbacca took on four storm troopers.



QUIET THINK/WORK TIME

Once I was certain that everyone understood the context and knew that their task was to determine the outcome of Round 3, I sent volunteers back to their seats and gave students the **activity sheet**. I instructed them to first work quietly on their own to consider the problem and start to work on a plan for determining which team would win Round 3. I suggested that even if they were not sure how to begin, they could start by deciding which team (the rebel fighters or the storm troopers) was stronger. That way, when it was time to begin working with their group, all students would have something to contribute to the conversation.

PUTTING STUDENTS IN GROUPS

After about five minutes of quiet work time, I passed out playing cards to make working groups. I put students in groups of two or three, thinking that groups of four might not allow for all students to have input into the group's solution strategy and to engage in SMP 3. Before they began to work together toward a final solution, I told each student to share his or her current thinking. I suggested that they start by saying, "I think that . . . because . . . ," even if they still weren't 100 percent sure who would win. This allowed everyone to contribute to the discussion because the expectation was to share thinking, not present a solution.

As soon as the students got into their groups, I was pleased to hear mathematical conversations begin. Most groups quickly decided that the team with Chewbacca and the three rebel fighters was stronger than the team with the four storm troopers, although many did not yet have a clear rationale other than the fact that Chewbacca was "really strong."

A handful of students initially ignored the information provided by Round 1, that the five rebel fighters were equal in strength to the four storm troopers. Instead, they used the information from Round 2 to erroneously decide that since Chewbacca was equal to two rebel fighters and one storm trooper, the storm troopers and rebel fighters were all equal in strength. Although this would allow them to arrive at the correct solution to the question of which team would win Round 3, it did not allow for a precise answer that indicated how much stronger the team of rebel fighters and Chewbacca was compared with the team of storm troopers.

STUDENTS' STRATEGIES: ASSIGNING VALUES

Several groups decided to give the characters a value, or "strength points," whereas others determined the outcome by using what I call "intuitive substitution." Among the students who awarded a point value, most of them started by giving each rebel fighter 1 point and were eventually able to use that value to decide that each storm trooper would be worth 1 1/4 points because five rebel fighters were equal in strength to four storm troopers. (One group needed a bit of help with this idea, so I pressed them to consider what the values of the two teams would be if each character were worth only 1 point as they originally thought. They determined that each storm trooper would have to be worth more than each rebel fighter, and at first they thought that the storm troopers would be worth 1 1/5 points. I had them add these, and when they determined that four 1 1/5s did not equal 5, they figured out that each storm trooper was worth 1 1/4 points.) Once the values were established, they determined that Chewbacca was worth 3 1/4 points; using this information, students were able to easily find that the rebel fighters and Chewbacca would win Round 3 by adding the

FRINGE THOUGHT: A person buying fertilizer for the lawn was surprised to find it only comes in one scent.

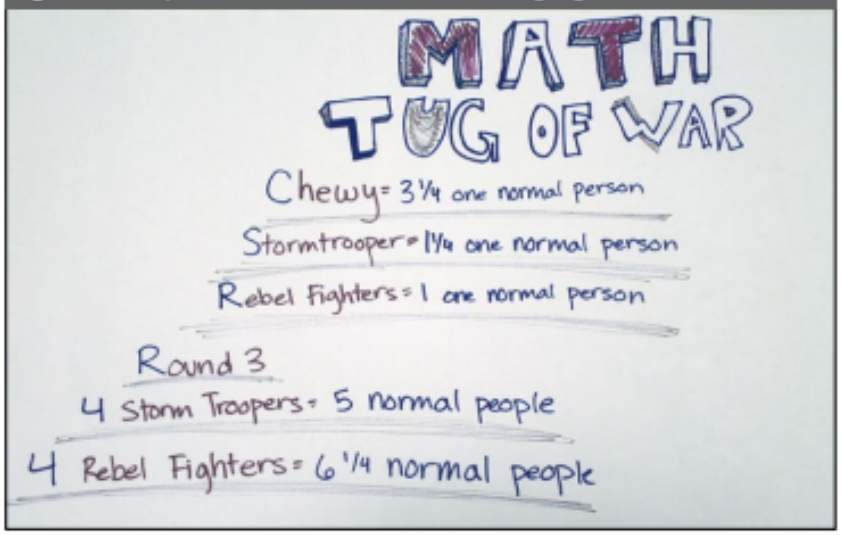
values of all of the participants and determining that 6 1/4 > 5 (see **fig. 5**).

Other students started with the idea that in Round 1 (five rebel fighters versus four storm troopers), each side had to be equal to 1. With this in mind, they determined that each rebel fighter had a value of 1/5 and each storm trooper had a value of 1/4. Students who chose this method were able to use the information from Round 2 to conclude that Chewbacca had a value of 13/20. They then added the values of all the participants and showed that 1 1/4 (the value of Chewbacca and three rebel fighters) was greater than 1 (the value of the four storm troopers).

Using the results of Round 1, one final group decided that, to begin with, each side would be worth 40 points. They explained this choice of starting value by stating that "both four and five can go into 40." They used tape diagrams to determine that each rebel fighter had a value of eight (since 40 ÷ 5 = 8), and each storm trooper had a value of ten (since 40 ÷ 4 = 10). Their strategy is shown in **figure 6**, in which the top bar is partitioned into five sections for the rebel fighters and the bottom bar into four sections for the storm troopers.

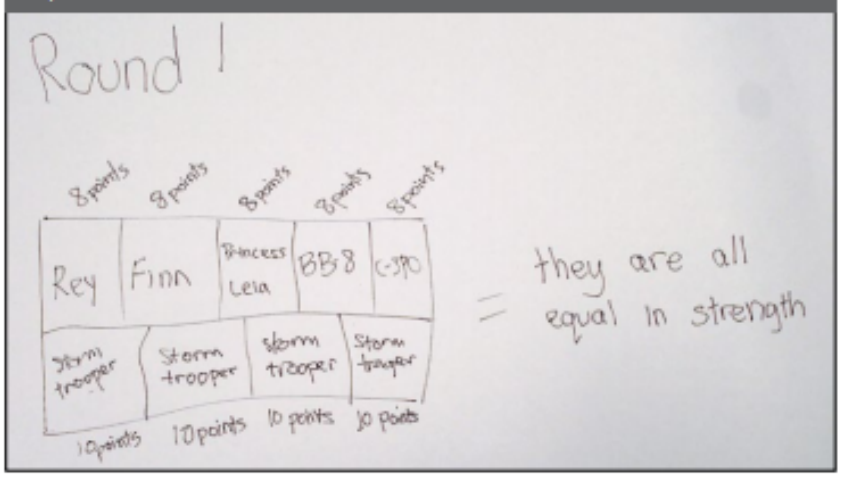
As described above, this group was then able to use these values to easily determine that Chewbacca was worth 26 points, and hence the rebel fighters won Round 3. Students assigning values were clearly engaged in SMP 2 and SMP 4 as they considered the problem in the context of a tug-of-war as well as a mathematical inequality.

Fig. 5 This sample of student work shows students assigning values.



HALEY KLEINE

Fig. 6 Students' tape diagram showed values assigned to rebel fighters and storm troopers.



HALEY KLEINE

STUDENTS' STRATEGIES: "INTUITIVE SUBSTITUTION"

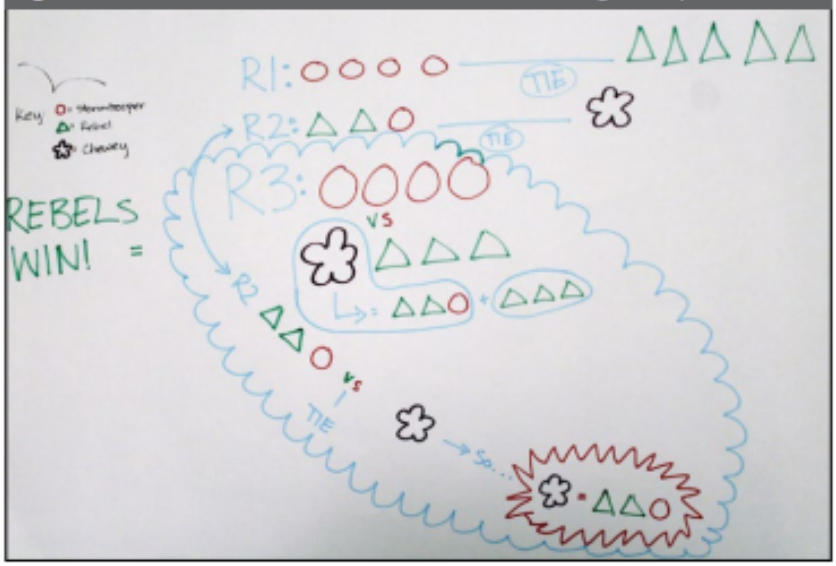
A few students were able to correctly find the results of Round 3 without needing to assign any numerical values to the characters. These students used what I call "intuitive substitution." I use the term "intuitive" because the students had not had any formal instruction on using substitution, so their use of it was strictly because it made sense to them. (An example is shown in **fig. 7**).

An explanation of this group's work may make it easier to understand:

- Round 1 (R1) is shown by the four red circles representing the four storm troopers versus five green triangles representing the five rebel fighters.
- Round 2 (R2) is shown by the two green triangles (two rebel fighters) and one red circle (one storm trooper) versus the purple cloud representing Chewbacca.
- Round 3 (R3) is shown by the four red circles (four storm troopers) versus one purple cloud (Chewbacca) and three green triangles (three rebel fighters). Chewbacca is equal in strength to

two rebel fighters and one storm trooper—just below the purple cloud representing Chewbacca, the group has drawn two green triangles and one red circle + three more green triangles. This is evidence of this group's use of structure (SMP 7) as they "see complicated things . . . as single objects or as being composed of several objects." Now the two teams in Round 3 are made up of five rebel fighters plus one storm trooper versus four storm troopers. Since Round 1 told us that five rebel fighters were equal in strength to four storm troopers, the side with the additional storm trooper wins.

Fig. 7 Some students used intuitive substitution to solve the tug-of-war problem.



HALEY HUENE

SHARING STRATEGIES AND PROCESSING MATHEMATICS

After all groups had completed recording their solutions and strategies, I had students move their chairs into a semicircle to present their work. I wanted the students to understand that when their classmates were presenting, they were to pay close attention to the speakers; likewise,

FRINGE THOUGHT: People with coughs don't go to the doctor. They go to a concert.

FRINGE THOUGHT: Lost time is never found again. –Benjamin Franklin

A few students were able to correctly find needing to assign any numerical values to accessing "intuitive substitution": Without they used what made sense to them.

when they were the presenters, they were to speak clearly and loudly enough for their classmates to hear. This provided another opportunity for students to engage in SMP 3.

We began with groups that assigned values to the characters. As groups shared their solutions, I created a chart (similar to **table 1**) showing the values attributed to each and the resulting contests in Round 1, Round 2, and Round 3.

Placing the different values in the same table allowed for a very productive conversation. Students noticed that strategies that initially looked different, such as assigning different values to the characters, were in fact the same as long as the relationships between the values were consistent. We did not address the equivalent ratios explicitly in this lesson, but the chart sets up the class nicely for this investigation in the future. The table also allowed the group that deemed the rebel fighters to each be worth 1/5 of a point and the storm troopers to each be worth 1/4 of a point to "attend to precision" (SMP 6) and revise their values for Round 3. They agreed with the other groups regarding which team won Round 3, but they realized that one of their final values (1 1/10) was incorrect.

REFLECTING ON THE TASK

As I wrote previously, this has always been a favorite task of mine, and this version did not disappoint! The students were instantly engaged, even though it can hardly be described as a "real-world" problem. Not only were they on task during the two periods we spent on the task, on the Friday following the introduction of the task, two students showed up wearing Star Wars shirts. Although this could have been a coincidence, I chose instead to believe that it was an indication of how much they enjoyed the challenge and the context.

The tug-of-war, whether in its original form or modified to reflect pop culture, has many benefits. First and foremost, it is a great example of a task that lends itself to multiple strategies and multiple approaches within the same strategy. Students used two main strategies—assigning values and substitution. Within the groups that assigned values, three different approaches were shared. Students used equations, symbols, and tape diagrams, thus engaging in SMP 4—model with mathematics. In addition, the tug-of-war problem sets up the class for a natural segue into the area of ratios and proportional relationships—a focal, and often challenging, domain in the middle school standards. Second, the problem does not initially look like math. As presented, there are no numbers involved with the exception of the number of characters; thus, it does not have the effect of turning students off or causing them to shut down, as they might if the problem looked more "mathy." The values and strategies that students used all came directly from them. They were never told, "This is an algebra problem that should be solved by assigning values or using substitution." Finally, the problem encouraged students to use substitution intuitively. Once students understood that they had equivalent values to work with, it made complete sense to use substitution to solve the final round.

As teachers take on the challenge

URFINGUSSTHINKSTOCK

Table 1 Student groups assigned values to the tug-of-war characters.

Rebel Fighter	Storm Trooper	Chewbacca	Round 1	Round 2	Round 3
1 point	1 1/4 points	3 1/4 points	5 = 5	3 1/4 = 3 1/4	6 1/4 > 5
1/5 point	1/4 point	13/20 point	1 = 1	13/20 = 13/20	1 1/10 > 1
8 points	10 points	26 points	40 = 40	26 = 26	50 > 40

the results without the characters by formal instruction,

of providing their students with rich mathematical tasks that provide opportunities for engagement in the Standards for Mathematical Practice (CCSSI 2010), I encourage them to revisit some favorites from the past. Many tasks, like the Mathematical Tug-of-War, are ready to use as is, although it never hurts to call on Chewbacca and the rest of the rebel fighters to help solve problems, intergalactic or otherwise.

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Let's Chat about a Mathematical Tug-of-War

On Wednesday, September 20, 2017, at 9:00 p.m. ET, we will expand on "Return of the Tug-of-War" (pp. 40–48), by Julie McNamara. Join us at #MTMSchat.

We will also Storify the conversation for those who cannot join us live. The *MTMS* monthly chats fall on the third Wednesday of the month.

Julie McNamara, julie.mcnamara@csueastbay.edu, teaches mathematics methods to aspiring teachers at Cal State East Bay in the San Francisco Bay Area. She also takes advantage of opportunities to work with students in K–12 classrooms any chance she gets.

FRINGE THOUGHT: To disagree, one doesn't have to be disagreeable.

activity sheet

Name _____

STAR WARS TUG OF WAR

Your task: Determine who will win the final tug-of-war (Round 3). The first two rounds will give you the information you need. You may want to make a sketch in the space below each round.

Round 1:
On one side are five rebel fighters: Rey, Finn, Princess Leia, BB-8, and R2-D2. Surprisingly, **they are all of equal strength.**

On the other side are four storm troopers. **They, too, are all equal in strength.**

In the contest between these two teams, the result is dead even. **Neither team can outpull the other.**

Round 2:
On one side is Chewbacca, who is known around the galaxy for being very strong.

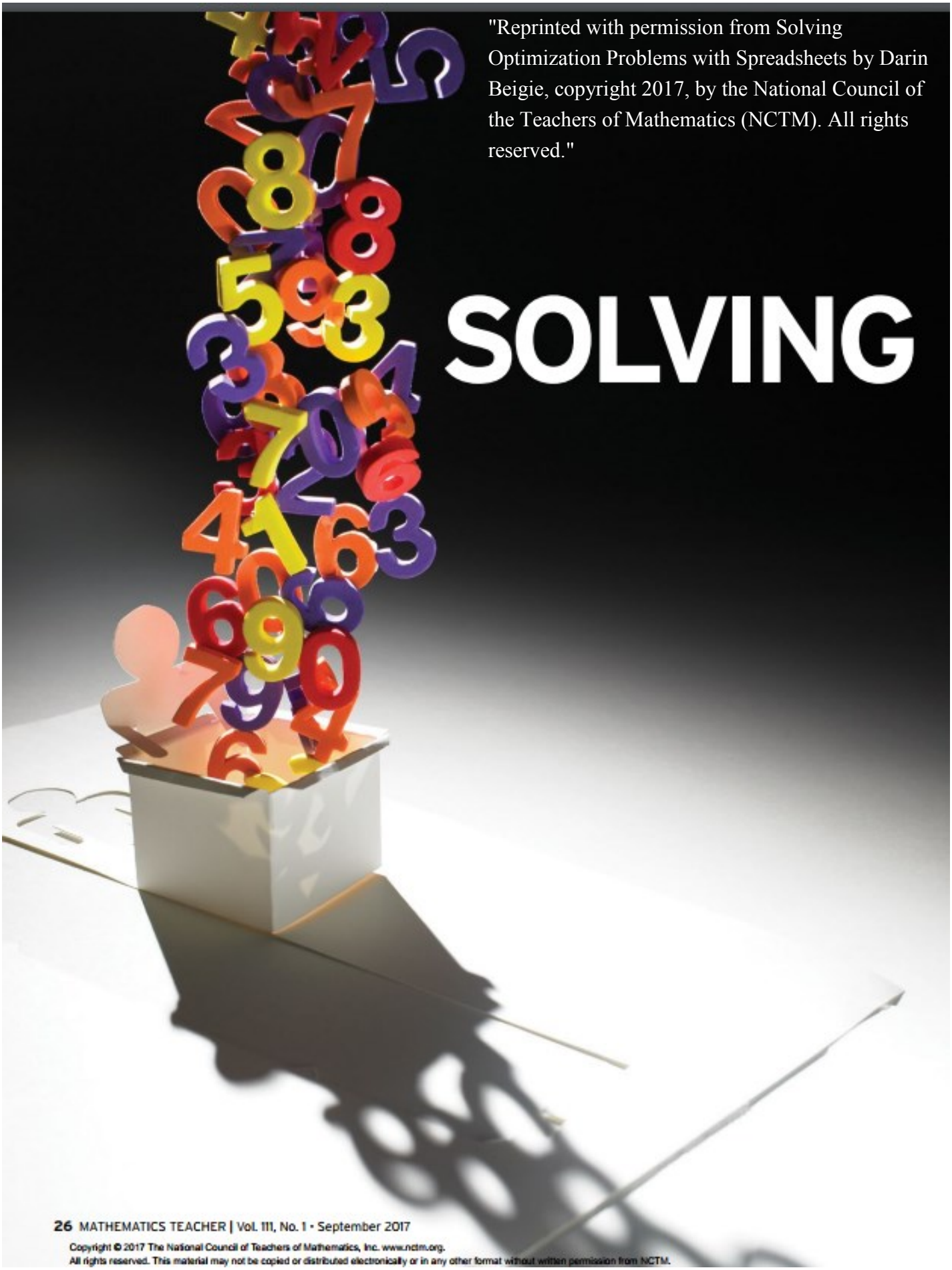
On the other side is a team made up of Rey, Finn, and one storm trooper. **Again, it's a draw—an equal pull.**

Round 3:
It's the final tug that you must figure out.

It will be between these two teams: Chewy, Rey, Finn, and Princess Leia on one side and the four storm troopers on the other side.

Can you figure out who will win this tug-of-war?

from the September 2017 issue of mathematics teaching in the MIDDLE SCHOOL



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FRINGE THOUGHT: A child is a strange creature who wants to find out everything by asking questions no one can answer.

OPTIMIZATION PROBLEMS WITH SPREADSHEETS

These classic activities usually reserved for calculus are recast in a highly visual approach as investigations with first-year algebra students.

Darin Beigie

Spreadsheets provide a rich setting for first-year algebra students to solve problems. Individual spreadsheet cells play the role of variables, and creating algebraic expressions for a spreadsheet to perform a task allows students to achieve a glimpse of how mathematics is used to program a computer and solve problems. Classic optimization problems that are usually reserved for calculus can be recast as a spreadsheet investigation with younger students. The result is a highly visual approach to the topic of optimization, where emphasis is placed on analyzing and interpreting results. Studying how to make a situation optimal, such as maximizing profit or minimizing travel time, offers algebra students purposeful questions that have real-world relevance.

We discuss four problems to show how accessible classic optimization problems can be for younger students. The spreadsheet approach is robust, and the possibilities are as varied and endless as the more traditional algebra word problems found in textbooks. Introduction to the mechanics of using a spreadsheet is minimal. Students need to know how to refer to a cell location and enter a formula, which we briefly demonstrate at the beginning of the activity. Throughout the year, students learn how to create algebraic expressions to describe patterns and solve word problems, so they are receptive to the idea of creating algebraic expressions to make spreadsheet formulas.

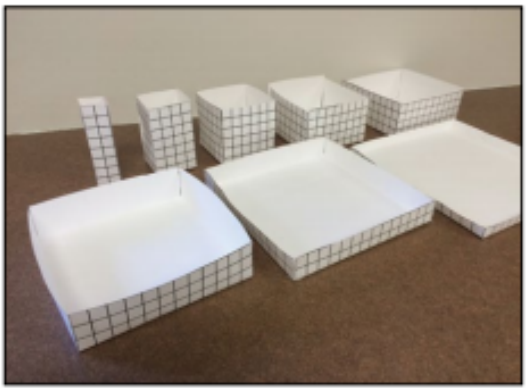
Spreadsheet activities have been incorporated in math curricula for a wide range of grade levels and topics, from data analysis and patterns for elementary school children (Ploger et al. 1997) to topographical maps and recursive processes for high school students (Feicht 1999; Burke 2012). The benefits of spreadsheet use in math classrooms have included greater connection with real-world problems (Niess, Sadri, and Lee 2008), use of numerical modeling to promote STEM-related topics (Benacka 2016), opportunities for more student-centered learning (Agyei and Voogt 2016), and enhanced tools for data analysis (Beigie 2010) and problem solving (Lesser 1999; Engerman, Rusek, and Clariana 2014). Use of spreadsheets with elementary school children (grades 3–5) has been encouraged for many years now (NCTM 2000, p. 207),

and there has been recognition that spreadsheets can help students in the early stages of algebra (middle school and early high school) make a more authentic transition from working with numbers to creating variables (Friedlander 1998). With these optimization problems, we hope to foster a much broader view of what solving a word problem can mean for algebra students.

Optimization problems in a financial context offer young students exposure to some of the most fundamental principles of economics.

EXPANDING THE NOTION OF AN ALGEBRA WORD PROBLEM

As with traditional word problems, there is flexibility with how spreadsheet activities are incorporated into an algebra course. We do about one spreadsheet activity per month, generally tackling more sophisticated problems as students' expertise with creating algebraic expressions grows. We introduce each activity with a brief but open class discussion about real-world context, such as the motive to maximize profit or minimize travel time. Students work side-by-side in pairs (each with his or her own laptop) to construct the spreadsheet solution. Working in pairs helps students troubleshoot any issues that arise with constructing their spreadsheets and allows them to collaborate on creating formulas. We generally expect students to do pencil-and-paper work generating algebraic expressions before they enter formulas into their spreadsheets, but as the year progresses, students find it increasingly natural to go directly to entering spreadsheet formulas on their computers. Students comfortably finish constructing the spreadsheet by the end of the lesson, but it is not unusual for the written analysis to extend into that night's homework. No prior knowledge of spreadsheets is assumed, so the opening activity is prefaced by direct instruction on creating a spreadsheet solution (as described in the **appendix**; see online at **http://www.nctm.org/mt**). With later activities, students quickly become proficient with the mechanics of spreadsheet construction, and the emphasis is on creating formulas and analyzing spreadsheet calculations.



DARIN DECKER

Fig. 1 Students' open boxes have different sizes, depending on the size of the corner squares that they cut from the original 18 in. by 18 in. sheet of paper.

Solving optimization problems in algebra provides opportunities to foster new thinking and broaden the notion of what it means to solve an algebraic word problem, phrased in the context of the Common Core's (CCSSI 2010) standards for mathematical practice (SMP) and content standards.

- Instead of creating algebraic equations to solve word problems with formal algebraic steps (A-CED.1–3; A-REI.1–7), students create algebraic expressions to program a computer to solve optimization problems.
- In addition to focusing on linear, quadratic, and exponential models to solve problems (F-LE.1–3), students study a wide variety of nonlinear relationships that arise naturally in real-world settings.
- Reasoning abstractly and quantitatively (SMP 2) and constructing viable arguments (SMP 3) includes particular emphasis on having students provide verbal and written interpretation of their calculations.

We describe four spreadsheet activities, sampled in chronological order from the sequence of optimization problems studied throughout the school year. Three activities are described here; a fourth is online at **http://www.nctm.org/mt**.

MAXIMIZING THE VOLUME OF AN OPEN BOX

Start with an 18 in. by 18 in. square piece of paper; cut out squares x inches in length from the corners, and then fold up the resulting sides to form an open box. For what value of x will the volume of the box be the largest?

PREVIOUS PAGE: PHIL ASHLEY/THINKSTOCK

FRINGE THOUGHT: Another reason you can't take it with you—it goes before you do.

Each student is given an 18 in. by 18 in. piece of paper with a square inch grid on one side and assigned a different size of square to cut from the corners, ranging from 1 in. by 1 in. to 8 in. by 8 in. (in whole-number increments). Students fold up and tape the resulting sides to form an open box (see **fig. 1**). The full spectrum of all the boxes is displayed, and we have a class discussion about the box volumes. Different boxes clearly have different volumes; for example, the box resulting from cutting 8 in. by 8 in. corners is thin (albeit the tallest) and has less volume than the box resulting from cutting out 4 in. by 4 in. corners.

Students work on a table that models the intended spreadsheet construction (see **fig. 2**). They create algebraic expressions for the length, width, and height of the open box, and then use their expressions to write spreadsheet formulas in the table. They deduce that the height of the open box equals the length of the removed squares, x inches, and the base of the open box is a square with length $18 - 2x$ inches. As described in the appendix, students enter an arbitrary square length in cell A2 and appropriate formulas in cells B2 through D2, then use the *Fill* and *Chart* options to have the spreadsheet create a table and corresponding graph relating the open box volume to the length of the removed squares (see **fig. 3**). We choose a scatter plot with smooth lines and markers, instead of a bar graph, to emphasize that underlying the discrete ordered pairs generated by the spreadsheet is a continuous relationship defined by a formula.

Along with student construction of the open boxes, the spreadsheet graph provides a strong visual of the optimization process. Students use the spreadsheet table and graph to determine the maximum volume of the open box and then draw a sketch of the optimal box dimensions (several examples of student work on all these problems are online with this article at **http://www.nctm.org/mt**). More important, we ask students to describe in words what is happening in the optimization problem, to explain why the volume increases, peaks and then decreases again as the box height increases. On the basis of their box constructions, students recognize that the box volume is largest when the height is neither too small nor too large. A more subtle point is why the optimal height is closer to 0 inches than to 9 inches, and some students try to verbalize their reasoning on this point. For example, Andrew writes, "If we pick a smaller height, the L and W will be greater. So, although picking 3 as a height seems a little small, it is OK because the length and width will be 12. It's okay to sacrifice some height for greater length and width."

height=x length=width=18-2x

	A	B	C	D
1	Height (in.)	Length (in.)	Width (in.)	Volume (in.³)
2	#	=18-2*A2	=B2	=A2*B2*C2

Fig. 2 Jason (pseudonyms are used throughout) makes algebraic expressions and then creates spreadsheet formulas for the Open Box problem.

	A	B	C	D
1	Height (in.)	Length (in.)	Width (in.)	Volume (in.³)
2	0	18	18	0
3	1	16	16	256
4	2	14	14	392
5	3	12	12	432
6	4	10	10	400
7	5	8	8	320
8	6	6	6	216
9	7	4	4	112
10	8	2	2	32
11	9	0	0	0

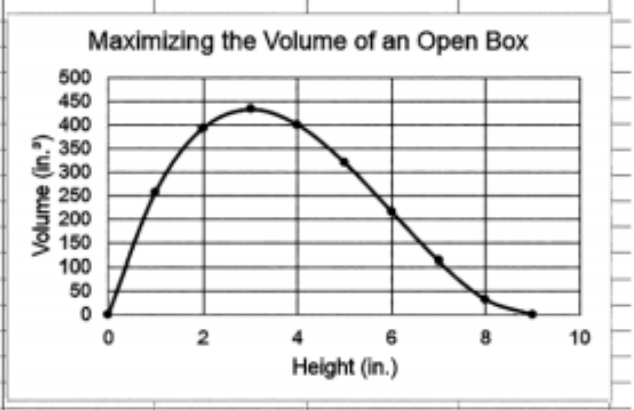


Fig. 3 The Open Box spreadsheet shows a maximum volume when the box height is around 3 inches.

The analysis fosters the global perspective of a graph as a collection of all solutions (A-REI.10), with each ordered pair corresponding to a different open box. Students recognize that the intercepts represent two very different nonoptimal limiting cases: the points near (0, 0) represent a box with small volume because the removed corners, and resulting box height, are very small; the points near (9, 0) represent a box with a small volume because the removed corners are very large and the resulting base is very small. Students are asked to locate their constructed box on their spreadsheet graph, and they recognize that points along the graph correspond to the spectrum of different boxes they constructed as a class.

Many variations on the Open Box problem exist,

such as maximizing the volume of a rectangular box to be shipped by the U.S. Postal Service. According to the USPS website (https://www.usps.com/ship/preparing-domestic-shipments.htm), unless you use its retail ground service, it requires that parcels not exceed 108 in. in combined height and girth (where *girth* is defined as "twice the length plus twice the width of a rectangular box"). We have students find the maximum volume of a square-base box that can be shipped under the USPS constraint. A wealth of optimization problems involving two-dimensional geometry also exist, and another of our spreadsheet investigations involves minimizing the perimeter of a fence that encloses a rectangular garden of fixed area.

x = # pairs of shoes $\qquad 45 - 0.05x$ = price per pair

$(45 - 0.05x)\,x$ = income $\qquad 15x$ = expenses

	A	B	C	D	E
1	# of Pairs of Shoes	Price per Pair ($)	Income ($)	Expenses ($)	Profit ($)
2	#	=45-.05*A2	=B2*A2	=15*A2	=C2-D2

Fig. 4 Chris makes algebraic expressions and then creates spreadsheet formulas for the Shoe Profit problem.

MAXIMIZING PROFIT

The shoe company Stylin Mylin sends a weekly shipment to a local shoe store, The Friendly Foot. Each pair of shoes costs Stylin Mylin $15 to make. As an incentive for The Friendly Foot to order more shoes, Stylin Mylin offers them a price per pair of $45 minus 5¢ for every pair of shoes that they buy. For example, if The Friendly Foot orders 40 pairs of shoes, the price per pair would be $43. If The Friendly Foot orders 100 pairs, the price per pair would be $40, and so on.

Pairs of Shoes	Price per Pair ($)	Income ($)	Expenses ($)	Profit ($)
0	45	0	0	0
20	44	880	300	580
40	43	1720	600	1120
60	42	2520	900	1620
80	41	3280	1200	2080
100	40	4000	1500	2500
120	39	4680	1800	2880
140	38	5320	2100	3220
160	37	5920	2400	3520
180	36	6480	2700	3780
200	35	7000	3000	4000
220	34	7480	3300	4180
240	33	7920	3600	4320
260	32	8320	3900	4420
280	31	8680	4200	4480
300	30	9000	4500	4500
320	29	9280	4800	4480
340	28	9520	5100	4420
360	27	9720	5400	4320
380	26	9880	5700	4180
400	25	10000	6000	4000
420	24	10080	6300	3780
440	23	10120	6600	3520
460	22	10120	6900	3220
480	21	10080	7200	2880
500	20	10000	7500	2500
520	19	9880	7800	2080
540	18	9720	8100	1620
560	17	9520	8400	1120
580	16	9280	8700	580
600	15	9000	9000	0

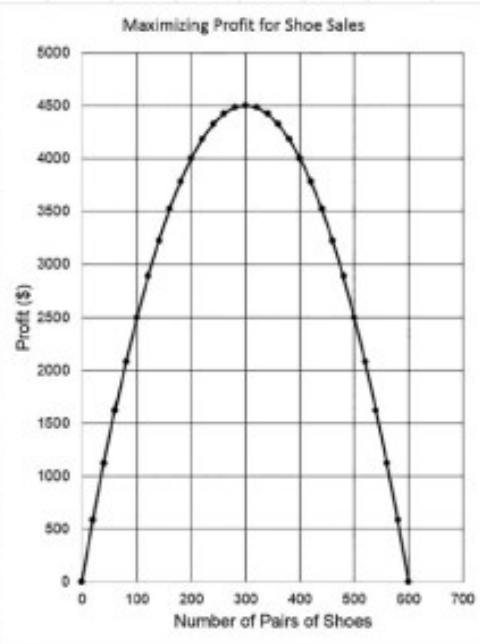


Fig. 5 The Shoe Profit spreadsheet shows a maximum profit when 300 pairs of shoes are shipped and the company must restrict the shipment to less than 600 pairs of shoes to obtain a positive profit.

DUBAS SYTHINKSTOCK

FRINGE THOUGHT: Things done by halves are never done right.

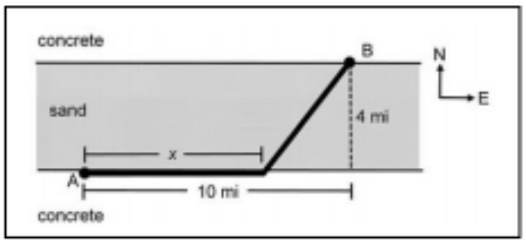
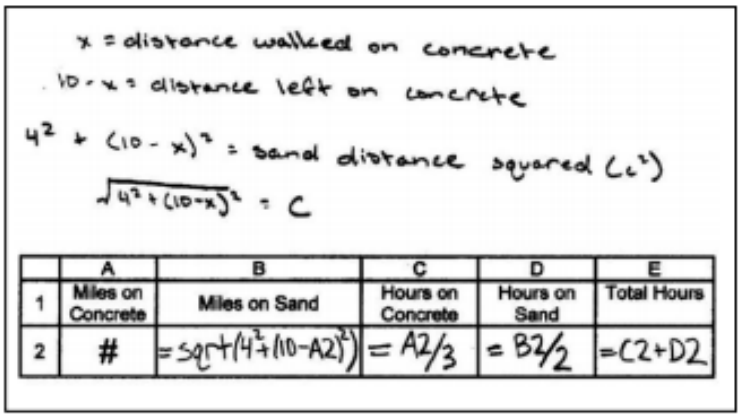


Fig. 6 The route along concrete and across sand is shown for the Travel Time problem.

What is the maximum profit that Stylin Mylin can make from the shipment? How many pairs of shoes must they ship to achieve this maximum?

Optimization problems in a financial context offer young students exposure to some of the most fundamental principles of economics. Lower prices incentivize people to purchase a product, so to maximize profit a company must seek to achieve the right balance between the opposing forces of getting the highest price from the largest number of people. As with the Open Box problem, students begin by writing algebraic expressions to create spreadsheet formulas (see **fig. 4**). With the second optimization problem, the mechanics of creating the spreadsheet becomes fairly automatic for students, requiring little teacher prompting: They enter an arbitrary number of pairs of shoes in cell A2 and appropriate formulas in cells B2 through E2, then use the *Fill* and *Chart* options to have the spreadsheet create a table and corresponding graph relating profit to number of pairs of shoes (see **fig. 5**). For the Open Box problem, the domain and increment to define the *Fill* procedure followed naturally from the student construction of the boxes: The height varied from 0 in. to 9 in. in increments of 1 in. In other problems, coming up with their own domain and increment is manageable for students, but it can be time-consuming and can vary greatly from one student to the next. We usually provide students with a particular choice to construct their table: 0 to 600 pairs of shoes in increments of 20 pairs for the profit problem. Once the spreadsheet graph is constructed, we expect students to understand and explain the choice: The 600 pairs of shoes represents the case where the shoe price is low enough that the income matches the expenses and the profit falls to zero.

As with the Open Box problem, the graph presents a strong visual context for interpreting the optimization. As the price per pair of shoes is lowered, the profit increases because more shoes are sold. However, eventually the profit peaks and decreases to zero as the lower price outweighs the increase in number of shoes sold. Interpretation



x = distance walked on concrete

$10 - x$ = distance left on concrete

$4^2 + (10 - x)^2$ = sand distance squared (c^2)

$\sqrt{4^2 + (10-x)^2} = C$

	A	B	C	D	E
1	Miles on Concrete	Miles on Sand	Hours on Concrete	Hours on Sand	Total Hours
2	#	=sqrt(4^2+(10-A2)^2)	= A2/3	= B2/2	=C2+D2

Fig. 7 Brady makes algebraic expressions and then creates spreadsheet formulas for the Travel Time problem.

and explanation of the optimization process continues to be an essential part of the analysis: Students must identify the maximum profit and explain why the profit increases, peaks, and goes back down to zero as the number of shoes increases. A natural consequence of the analysis is the need for the shoe company to place a limit on the number of shoes they sell to ensure profit.

The financial setting gives real-world context to interpreting complicated expressions (in this case, quadratic) by breaking them down into simpler pieces (A-SSE.1). Income is a product of the number of pairs of shoes, x, and price per pair of shoes, $45 - 0.05x$; and these two components oppose each other as x increases. More shoes sold is better for the company, but a lower price per pair is worse. The curved nature and peak of the graph are the result of these opposing forces. Understanding the domain of the *Fill* command provides a vivid opportunity to relate the domain of a function to its graph (F-IF.5): The profit remains positive until the number of pairs sold results in a price per pair that equals the cost to make the shoes.

MINIMIZING TRAVEL TIME

A person walks from A to B, where B is 10 miles east and 4 miles north of A (see **fig. 6**). He walks a certain distance on concrete, and the remaining distance on sand, as shown. His walking speed is 3 miles per hour (mph) on concrete and 2 mph on sand. How many miles should he walk on concrete to minimize the travel time from A to B?

Students use the Pythagorean theorem to create an algebraic expression for the number of miles walked on the sand, which equips them to create the spreadsheet formulas (see **fig. 7**). From the initial diagram in **figure 6**, students readily

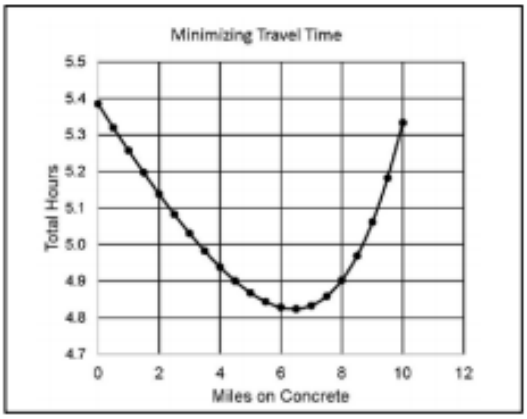


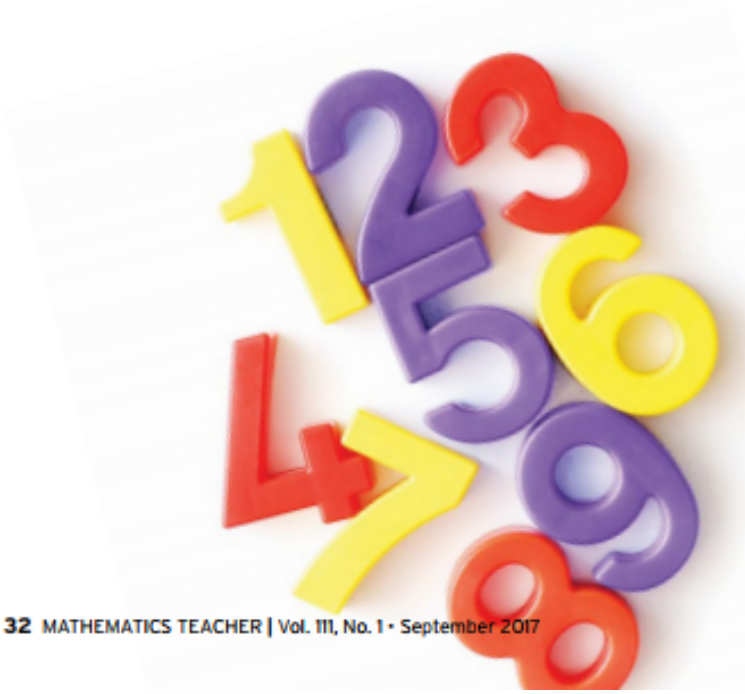
Fig. 8 The Travel Time graph shows the minimum travel time occurs when the distance traveled along concrete is about 6.5 miles. Further search with the spreadsheet pinpoints the concrete distance to 6.4 miles (rounded to the nearest tenth).

Spreadsheets offer a powerful tool that allows algebra teachers to expand the notion of what it means to use algebraic reasoning to solve a word problem.

understand that the first column (number of miles on concrete) varies from 0 miles to 10 miles, and we suggest increments of 0.5 miles to create the spreadsheet table and graph (see **fig. 8**).

The minimum travel time does not occur for a half-mile increment in the table, so further analysis is needed. In fact, for most optimization problems, students must supplement their table and graph by copying and pasting one of the rows from their table onto a blank row below the table. To pinpoint the final answer, students then adjust the column A number to refine their search near the suspected optimal solution. To the nearest one-tenth mile, they find that the minimum travel time that occurs when the person walks 6.4 miles along the concrete before pivoting and walking the rest of the way by sand. As students continue to analyze their solutions in different contexts, they see a common theme emerge: Optimization results from finding a balance between two opposing forces. If the person pivots too quickly onto the sand, he will take more time because of the slower speed in sand; if the person pivots too late onto the sand, he will take more time because of the overall longer distance he has to travel.

The refined search for minimal travel time has students attend to precision to solve the problem (SMP 6). As students become more accustomed to studying optimization problems, they become increasingly familiar with interpreting key features of graphs (F-IF.4). The endpoints correspond to the limiting (and least efficient) cases of walking the greatest and least distances on sand, and the minimum corresponds to the most efficient case that balances the distances on sand and concrete.



ALGEBRA AS THE LANGUAGE OF COMPUTERS

Traditional word problems understandably continue to be a part of first-year algebra curricula, but *Principles to Actions: Ensuring Mathematical Success for All* emphasizes the need for students to have a range of strategies and approaches from which to choose in solving problems (NCTM 2014). Spreadsheets offer a powerful tool that allows algebra teachers to expand the notion of what it means to use algebraic reasoning to solve a word problem. A wealth of optimization problems, usually reserved for calculus students, are made accessible to younger students in a highly visual manner. Seeking an optimal solution has as much real-world relevance and context as the more traditional use of algebraic steps to solve an equation. Making algebraic expressions to create spreadsheet formulas lies at the heart of algebraic thinking and

DUBASSY/THINKSTOCK

FRINGE THOUGHT: Be cautious. Opportunity does the knocking for temptation, too.

gives students the opportunity to see the essence of computing, in particular using mathematics to program a computer. Students generally show pride in workmanship with their written interpretations and explanations, and we feel this is the result of studying authentic and relevant problems. Indeed, we impress on our students that the spreadsheet investigations offer a powerful, STEM-related answer to the age-old student question, "When will I ever use this?" because millions of people across the globe use algebraic reasoning every day to create the programming code that runs our computers and devices.

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more4U

Student work, an additional activity problem, and an appendix are online with this article at **http://www.nctm.org/mt**. This more4U content, an additional benefit, is for members only.

Affiliate News

Laurel Highlands Mathematics Alliance (LHMA)

LHMA revived its Try-Math-a-Lot problem solving competition for sixth and seventh graders. It was held on May 10, 2017 at Central Cambria Middle School in Ebensburg, PA. The event was co-sponsored by the Mathematics Departments of University of Pittsburgh at Johnstown and St. Francis University. There were 104 students from 11 area schools that attended the event which consisted of three sessions ranging from "clip-clue" logic puzzles, "blocking the view" building structures, and a math quiz bowl.

Plans have begun for LHMA's Fourth Annual Math Educator Mini-Conference for K-12 teachers. It will be held October 21, 2017 at Indiana University of PA in conjunction with the PCTM's Pre-Service Teacher Day.

PAMTE

PAMTE will be conducting three virtual discussion groups for its members during Fall 2017 on the *AMTE Standards for Preparing Teachers of Mathematics.*

PAMTE's 12th Annual Symposium will be held *May 16-17, 2018* at Shippensburg University. AMTE Executive Director, Tim Hendrix, will provide the keynote address. He was originally scheduled to present at the 11th Annual Symposium but had to cancel due to unfortunate circumstances.

PAMTE is on Facebook! Our page is used to share information relevant to the mathematics education community. Check us out at https://www.facebook.com/groups/PAmatheduc/

PAMTE membership is open to all those interested in promoting and improving the education of pre-service and in-service teachers of mathematics across the Commonwealth of Pennsylvania. Our recently updated website now includes *online* membership registration. Visit http://www.pamte.org/

PA Mathematics Education Coalition Update

Kate Remillard

The PA Mathematics Education Coalition was instrumental in bringing PA's Secretary of Education Pedro Rivera to headline at the 66th Annual PCTM Conference on Thursday, August 3rd. Special thanks are extended to Marian Avery for coordination. The Secretary spoke of wanting to change the culture of education nationally and locally. He additionally encouraged conference attendees to do a better job "celebrating" classroom accomplishments. The Coalition hopes to continue to build a working relationship with PDE.

The Coalition additionally continued its *Act 82 of 2012* letter writing campaign at the PCTM Conference. Committee members staffed a table where attendees could address and sign a letter to their state legislators encouraging them to consider modifications to teacher evaluation. For those interested in participating, the form letter can be downloaded on the PCTM website, "members only" section.

The Pennsylvania Department of Education released its draft of the Consolidated State Plan pursuant the Every Student Succeeds Act (ESSA) on Wednesday, August 2nd and held a public comment period through August 31st. The Coalition developed and shared a number of recommendations for comment by mathematics educators based on research and national professional organization guidance. PA's final plan will be submitted to the U.S. Department of Education on September 18th.

FRINGE THOUGHT: Your work is a commentary on your character.

A Historical Statistical Journey Through Data, Chance, and Uncertainty

Kevin Robinson

Fall 2017 – As promised in my last column (Winter 2017), we will learn, in this column, who the original Mr. T was and why his work was so crucial in the development of the commonly used t-test for means. The original Mr. T, William Sealy Gosset, is associated with the discovery of the t-distribution and the resulting profound effect on the practice of statistics. The story of Gosset is one of my favorites to share with students, bringing into the curriculum personality and historical perspective.

I begin the class discussion by showing the picture of Gosset (see Figure 1) and asking, "Does anybody recognize this gentleman?" Typically, there is little discussion or a few comments about the mustache. Only once has a student recognized the gentleman from the Guinness TV commercials that used the phrase "Brilliant." See Figure 2. To see a video of a commercial, click here.

Figure 1. William Sealy Gosset (1876 – 1937).

Figure 2. Guinness TV commercial—brilliant campaign.

Indeed, Gosset was a brilliant scholar who studied both chemistry and mathematics at New College, Oxford. He joined the Guinness brewery in Dublin, Ireland upon graduation in 1899. He invented the t-test (1908) to handle small samples for quality control in brewing. He discovered the form of the t-distribution by a combination of mathematical and empirical work with random numbers, an early application of random simulation methods. See Figure 3.

Gosset published under the pseudonym "Student" for proprietary reasons, as the Guinness brewery was afraid of releasing trade secrets. In addition, the work of Gosset (McMullen, 1939; Pearson, 1939) opened the door for the t-procedures to play a role in a number of widely used statistical analyses, including testing and estimation of means and inference in linear regression analysis. A tribute described Gosset as follows:

> *To many in the statistical world 'Student' was regarded as a statistical advisor to Guinness's brewery, to others he appeared to be a brewer devoting his spare time to statistics. ... though there is some truth in both these ideas they miss the central point, which was the intimate connection between his statistical research and the practical problems on which he was engaged.* (McMullen, 1939, p. 207)

Student's *t*-distribution

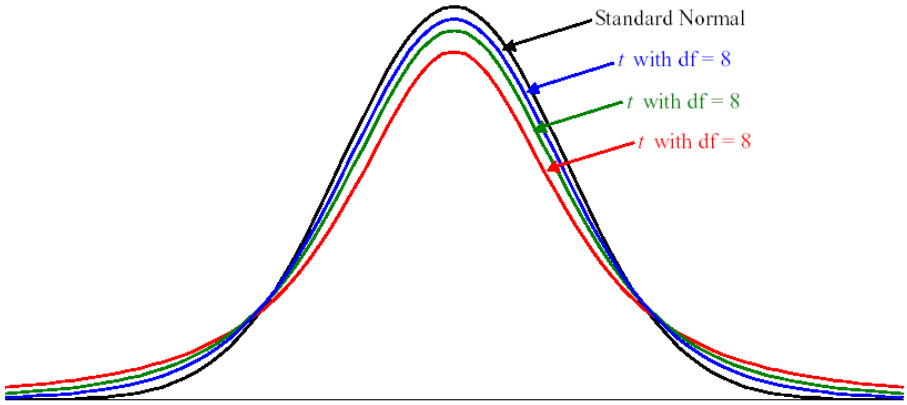

Figure 3. Student's t-distribution.

In the next column, we will introduce the individual most commonly called "The Father of Statistics." Until then, I wish you a safe and productive 2017-18 school year. Reminder: a number of resources, including a timeline of statistics, are at the following website: http://sites.millersville.edu/krobinson/STATHIST/. Statistically Yours ~ KSR

William Sealy Gosset, 1876-1937, in E. S. Pearson and M. G. Kendall,
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Kevin S. Robinson, PhD, is an Associate Professor in the Department of Mathematics at Millersville University of PA. He is a statistical educator with interest in K-16 statistical curriculum and the application of industrial statistics. An avid Pittsburgh sports fan, Kevin and his wife Becky are the proud parents of two sons, Ethan (13) and Caleb (9).

FRINGE THOUGHT: It is never too soon to be kind, for we never know how soon it will be too late.

Submissions Solicited for PCTM Magazine

Since the 1990's, the Pennsylvania Council of Teachers of Mathematics (PCTM) has produced the PCTM Magazine for our members. Our mission is to promote mathematics education in Pennsylvania. In the magazine we accomplish this by publishing edited articles by leading authors and local news from around the state. PCTM is committed to improving mathematics instruction at all levels. We place an emphasis on classroom activities that are aligned to the Pennsylvania Core State Standards and the NCTM Principles and Standards for School Mathematics.

You are invited to submit articles for consideration for publication in the PCTM Magazine. This publication provides an excellent opportunity for you to share your ideas with the ever-growing number of colleagues dedicated to improving mathematics education in Pennsylvania. Any topic of interest to teachers of mathematics, especially K-12 classroom teachers in Pennsylvania, is suitable subject material. All readers are encouraged to contribute articles and opinions for any section of the magazine. Teachers are encouraged to submit articles for Voices From the Classroom, including inspirational stories, exemplary lessons, or management tools.

Original artwork on the cover is another way teachers may contribute. We publish the magazine three times each school year, in the winter, spring, and fall.

Deadline for submissions:

Winter, December 15
Spring, April 15
Fall, August 15

Author Guidelines:

Manuscript Format: Manuscripts should be double-spaced, with 1-inch margins on all sides, typed in 12-point font and follow the APA 6th Edition style guide. Manuscripts should be submitted in Microsoft Word. If you have a picture or graphic in the text, please include the original picture(s) in a separate file. A cover letter containing author's name, address, affiliations, phone, e-mail address, and the article's intended audience should be included in the e-mail.

Manuscript Submission: Manuscripts should be submitted electronically as an e-mail attachment to pctm.editor@gmail.com. Receipt of manuscripts will be acknowledged. After review by the editors, authors will be notified of a publication decision.

Dates of Upcoming Conferences

NCTM Regional Conference, October 18-20, 2017, Orlando, FL

Preservice Teacher Day (for the Eastern part of PA) held in conjunction with the Careers in Mathematics Conference sponsored by EPaDel, October 21, 2017, Millersville University, Millersville, PA

Preservice Teacher Day (for the Western part of PA) held in conjunction with the 5th Annual LHMA Mini-Conference, October 21, 2017, Indiana University of PA, Indiana, PA

ATMOPAV Fall Conference October 28, 2017, Springside Chestnut Hill Academy, Philadelphia, PA

EPaDel (Eastern Pennsylvania Delaware section of the Mathematical Association of America) November 18, 2017, Shippensburg University, Shippensburg, PA

Joint Mathematics Meeting January 10-13, 2018, San Diego, CA

EPaDel (Eastern Pennsylvania Delaware section of the Mathematical Association of America) March 24, 2017, Temple University, Philadelphia, PA

NCTM Annual Meeting April 25-28, 2018, Washington, DC

PAMTE Annual Conference, May 16-17, 2018, Shippensburg, PA

PCTM Conference August 6-7, 2018, Harrisburg, PA